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DIRICHLET SERIES

S. MANDELBROJT

DIRICHLET SERIES

Principles and Methods

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To the memory
of Gladys

PREFACE

It is not our intention to present a treatise on Dirichlet series. This part of harmonic analysis is so vast, so rich in publications and in 'theorems' that it appears to us inconceivable and, to our mind, void of interest to assemble anything but a restricted (but relatively complete) branch of the theory.

We have not tried to give an account of the very important results of G. Pólya which link his notion of maximum density to the analytic continuation of the series, nor the researches to which the names of A. Ostrowski and V. Bernstein are intimately attached. The excellent book of the latter, which was published in the Collection Borel more than thirty years ago, gives an account of them with all the clarity one can wish for. Nevertheless, some scattered results proved by these authors have found their place among the relevant results, partly by their statements, partly as a working tool.

We have adopted a more personal point of view, in explaining the methods and the principles (as the title of the book indicates) that originate in our research work and provide a collection of results which we develop here; we have also included others, due to present-day authors, which enable us to form a coherent whole.

The reader will find chapters that are not connected with others. This is the case in Chapter IV, which is really only concerned with Taylor series, but is of a general character and, we believe, important enough to draw attention to.

Chapter I is elementary – it only contains the classical results that are indispensable for the techniques to be developed later.

Chapter X plays a special role: in the book itself (the first nine chapters) we do not give any bibliographical indications; it is Chapter X that provides the history of the results demonstrated in the preceding chapters, with a number in square brackets indicating the publication where the result in question is to be found.

Certain statements, for example part of those in Chapter V – a chapter devoted to theorems on the composition of singularities – reproduce in an improved form results contained in our *Rice Institute Pamphlet*, which has been out of print for several years.

As to the guiding ideas or principles which have served almost throughout the other chapters, we have to point first of all to what one could call the 'principle of arithmetic isolation' of the exponents, which consists in the following: the exponents whose fractional parts form a set that can be isolated by certain neighbourhoods around the fractional parts of the 'useful' exponents, neighbourhoods whose extent depends on the problem in question, provide autonomous information that cannot be destroyed by the other exponents; these exponents, which are isolated (by their fractional parts) from the others, already determine in a rather important manner the behaviour of the analytic continuation of the series.

Another principle consists, as we have already seen in other branches of analysis – by making the adherent series intervene – in inequalities concerning the coefficients. These inequalities use, when the exponents are given, the maximum of the function in a well-defined part of the plane; this part of the plane being defined as to its form or its dimensions (or both) by the arithmetic character of the exponents or by their growth.

I wish to thank Miss M. Caron and her collaborators at the Collège de France for the perfect way in which the typescript has been prepared.

April 1969 S. MANDELBROJT

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CHAPTER I

SEQUENCES OF EXPONENTS AND ASSOCIATED SEQUENCES. ELEMENTARY THEOREMS ON THE COEFFICIENTS AND ON CONVERGENCE

I.1. Ascending sequences of positive numbers

Let $\Lambda=\{\lambda_n\}$ $(n\geqslant 1)$ be a strictly increasing sequence of positive numbers: $0<\lambda_1<\lambda_2\ldots$ tending to infinity. The quantity

$$D = \limsup \frac{n}{\lambda_n}$$

is called the *upper density*[1] of Λ, and the quantity

$$h = \liminf\,(\lambda_{n+1} - \lambda_n)$$

the *step* of Λ.

If $h>0$, then $D^{\bullet}h\leqslant 1$, because for every ε with $0<\varepsilon<h$ there is an n_ε such that for $n>n_\varepsilon$ we have: $\lambda_n-\lambda_{n_\varepsilon}>(n-n_\varepsilon-1)(h-\varepsilon)$, hence

$$D^{\bullet} \leqslant (h - \varepsilon)^{-1}.$$

Denote by $N(x)$ the number of λ_n less than x $(x>0)$ and let $D(x)=N(x)/x$. Then $D(x)$ is the density function of Λ. We also have

$$D^{\bullet} = \limsup D(x).$$

The function

$$\bar{D}(x) = \frac{1}{x}\int_0^x D(y)\,dy \quad (x > 0)$$

is called the *function of mean density* of Λ, and the quantity

$$\bar{D}^{\bullet} = \limsup \bar{D}(x)$$

is the *upper mean density* of Δ.

It is easy to see that $\bar{D}^{\bullet}\leqslant D^{\bullet}$. But one can construct sequences for which $\bar{D}^{\bullet}<D^{\bullet}$. Here is an example with $D^{\bullet}=1$, $\bar{D}^{\bullet}=e^{-1}$: let $\{\nu_n\}$ be an ascending sequence of positive integers and set $\lambda_n=\nu_k+(n-\nu_{k-1})/(\nu_k-\nu_{k-1})$ for

$\nu_{k-1}<n\leqslant\nu_k$ $(k\geqslant 1, \nu_0=0)$. Here $D(x)<1$ $(x>0)$ and from $\lambda_{\nu_k}=\nu_k+1$ it follows that

$$D^{\cdot} \geqslant \lim \frac{\nu_k}{\nu_k+1} = 1,$$

that is, $D^{\cdot}=1$. Also, for $\nu_p\leqslant x<\nu_{p+1}$ $(p\geqslant 1)$ (since $\lambda_{\nu_p+1}=\nu_{p+1}+(\nu_{p+1}-\nu_p)^{-1}$) we have

$$D(x) \leqslant \frac{\nu_p}{x}$$

$$xD(x) = \sum_{k\leqslant p-1}\int_{\nu_k}^{\nu_{k+1}} D(y)\,dy + \int_{\nu_p}^{x} D(y)\,dy.$$

From these relations for $\nu_p\leqslant x<\nu_{p+1}$ and from $D(x)<1$ $(x>0)$ it follows that

$$\bar{D}(x) \leqslant \frac{1}{\nu_p}\left[\nu_1 + \sum_{k\leqslant p-1} \nu_k \log(\nu_{k+1}/\nu_k)\right] + \frac{\nu_p}{x}\log(x/\nu_p).$$

And since $\log x/x\leqslant 1/e$ for $x>0$, we can write

$$\bar{D}(x) \leqslant \frac{1}{\nu_p}\left[\nu_1 + \sum_{k\leqslant p-1} \nu_k \log(\nu_{k+1}/\nu_k)\right] + \frac{1}{e}.$$

If we choose,[2] for example, $\nu_n=E[\exp(\exp n)]$, then for $\nu_p\leqslant x<\nu_{p+1}$:

$$\bar{D}(x) \leqslant \frac{\nu_{p-1}(\log\nu_p - \log\nu_1)}{\nu_p} + \frac{\nu_1}{\nu_p} + \frac{1}{e},$$

that is,

$$\limsup \bar{D}(x) = \bar{D}^{\cdot} = e^{-1} < 1 = D^{\cdot}.$$

Observe that always $D^{\cdot}\leqslant e\bar{D}$ (this remark is due to Deny and Dvoretzky). In fact, since $D(x)\geqslant n/x$ for $x>\lambda_n$, we can write

$$\bar{D}(x) \geqslant \frac{1}{x}\left[\int_0^{\lambda_n} D(y)\,dy + n\log(x/\lambda_n)\right],$$

and by choosing $x=e\lambda_n$ we obtain

$$\bar{D}(e\lambda_n) \geqslant \bar{D}(\lambda_n)/e + n/(e\lambda_n) \geqslant n/(e\lambda_n),$$

that is,

$$\begin{aligned}\bar{D}^* = \limsup \bar{D}(x) &\geqslant \limsup \bar{D}(e\lambda_n)\\ &\geqslant \limsup n/(e\lambda_n) = D^*/e.\end{aligned}$$

The preceding example shows that in this inequality e cannot be replaced by a smaller constant.

We also see that D^* and $\bar{D}^*$ are simultaneously equal to zero, finite, or infinite.

If $D^* < \infty$, then the products $(n > 1)$

$$P_n = \prod_{m \neq n} |\lambda_n^2 - \lambda_m^2|/\lambda_m^2$$

converge; $P_n > 0$. The convergence of the product is an immediate consequence of the fact that $\lambda_n > kn$ with $k > 0$. The sequence $\{\Lambda_n\}$, where $\Lambda_n = P_n^{-1}$, is said to be *associated to* Λ.

THEOREM I.1.1. *There is a positive function* $A(a)$, *defined and finite for* $a > 1$, *such that for* $h > 0$

$$\limsup \Lambda_n/\lambda_n \leqslant A(a)D^* - 2a \log (hD^*) \cdot D^* = L(a; h, D^*).$$

When $D^* = 0$, *the expression* $L(a; h, D^*)$ *must be taken to be zero.*

For $A(a)$ we can choose the following function:

$$A(a) = a\{4 + \log [(a^2 - 1)/a^2] + \log [(a + 1)/(a - 1)]/a\}.$$

By taking, for example, $a = 3/2$ we have: $2a = 3$ and $A(a) < 8.5$.

Thus, in particular:

$$\limsup \Lambda_n/\lambda_n \leqslant [8.5 - 3 \log (hD^*)]D^*.$$

If m is a positive integer and p is fixed, we set $\mu_n = \lambda_{p+n}$ $(n \geqslant 1)$ and write:

$$\rho = \rho_{m,p} = \min_{0 \leqslant n \leqslant N} (\mu_{n+1} - \mu_n), \quad (\mu_0 = 0),$$

where $N = N_p(a\mu_m)$, $a > 1$ being fixed and $N_p(x)$ being the number of μ_n less than x. In other words, for $x > \lambda_{p+1}$ we have $N_p(x) = N(x) - p$, and for $x \leqslant \lambda_{p+1}$ we have $N_p(x) = 0$.

We denote by $\{M_n\}$ the sequence associated to $\{\mu_n\}$ and set

$$Q_m = A_m B_m,$$

where in a simplified notation with $\mu_m = \mu$:

$$A_m = \prod_{\substack{q \leqslant N \\ q \neq m}} \left| 1 - \frac{\mu^2}{\mu_q^2} \right|$$

$$B_m = \prod_{q > N+1} \left| 1 - \frac{\mu^2}{\mu_q^2} \right|.$$

Now

$$\begin{aligned}
\log A_m &= \sum_{\substack{q \leqslant N \\ q \neq m}} \log \left| 1 - \frac{\mu^2}{|\mu + (\mu_q - \mu)|^2} \right| \\
&\geqslant \sum_{k \leqslant m-1} \log \left[\left(\frac{\mu}{\mu - k\rho} \right)^2 - 1 \right] \\
&\quad + \sum_{k \leqslant N-m} \log \left[1 - \left(\frac{\mu}{\mu + k\rho} \right)^2 \right].
\end{aligned}$$

Since the function $1 - [\mu/(\mu + x\rho)]^2$ increases for $x > -\mu/\rho$, if we bear in mind that $-\mu/\rho < 1 - m$, we see that

$$\begin{aligned}
\log A_m &\geqslant \int_{1-m}^{0} \log\,[(\mu/(\mu + \rho x))^2 - 1]\,dx \\
&\quad + \int_{0}^{N-m} \log\,[1 - (\mu/(\mu + \rho x))^2]\,dx \\
&= \int_{1-m}^{N-m} \log\,[\rho|x|(2\mu + \rho x)/(\mu + \rho x)^2]\,dx \\
&\geqslant \rho^{-1} \int_{\rho(1-m)}^{\rho(N-m)} \log\,[|t|/(\mu + t)]\,dt \\
&\geqslant (N - 1)[\log \rho - \log\,(\mu + (N - m)\rho)] \\
&\quad + \rho^{-1} \int_{\rho(1-m)}^{\rho(N-m)} \log\,(|t|/\mu)\,dt.
\end{aligned}$$

As $N\rho = N_m(a\mu)\rho \leqslant a\mu$, we have $|t| \leqslant a\mu$ for $|t| \leqslant \rho N$. Also

$$\log \left[1 - \frac{(N - m)\rho}{\mu} \right] \leqslant \frac{(N - m)\rho}{\mu},$$

which allows us to write (taking account of the inequality $N\rho \leqslant a\mu$):

$$\begin{aligned}\log A_m &\geqslant 2\rho^{-1}\int_0^{\rho N}\log(t/a\mu)\,dt\\ &\qquad + (N-1)\log a - N(N-m)\rho\mu^{-1}\\ &\geqslant 2a\mu\rho^{-1}\int_0^{\rho\mathfrak{D}(a\mu)}\log\tau\,d\tau\\ &\qquad + (N-1)\log a - a(N-m)\\ &= [2\log\{\rho\mathfrak{D}(a\mu)\}\\ &\qquad - (2 + a - \log a)]N + ma - \log a,\end{aligned}$$

where $\mathfrak{D}(x)$ is the density function of $\{\mu_n\}$.

On the other hand, when we integrate by parts, we find

$$\begin{aligned}\log B_m &\geqslant \int_{a\mu}^{\infty}\log(1-\mu^2/x^2)\,dN_m(x)\\ &= -\log(1-a^{-2})N_m(a\mu) - 2\mu^2\int_{a\mu}^{\infty}\frac{\mathfrak{D}(x)}{x^2-\mu^2}\,dx\end{aligned}$$

for from $D^{\cdot} < \infty$ we have $N_m(x) = O(x)$ $(x \to \infty)$ (evidently the sequences $\{\mu_n\}$ and Λ have the same upper density $D^{\cdot}$) and

$$\lim\log(1-\mu^2/x^2)N_m(x) = 0.$$

Consequently

$$\begin{aligned}\log B_m &\geqslant -\log(1-a^{-2})N - 2\mu^2\mathfrak{D}^{\cdot}(a\mu)\int_{a\mu}^{\infty}\frac{dx}{x^2-\mu^2}\\ &= -\log(1-a^{-2})N + \mu\mathfrak{D}^{\cdot}(a\mu)\log[(a-1)/(a+1)],\end{aligned}$$

where

$$\mathfrak{D}^{\cdot}(x) = \sup_{y\geqslant x}\mathfrak{D}(y).$$

$$\begin{aligned}\log Q_m &\geqslant [2\log(\rho\mathfrak{D}(a\mu)) - (2 + a - \log a)]N\\ &\qquad - \log(1-a^{-2})N + \mu\mathfrak{D}^{\cdot}(a\mu)\log[(a-1)/(a+1)].\end{aligned}$$

Now the following relations are obvious:

$$\lim_{n\to\infty}\log(\Lambda_{n+p}/M_n)/\mu_n = 0,\ \lim_{p\to\infty}\rho_{m,p} = h; \qquad \lim\mathfrak{D}^{\cdot}(x) = D^{\cdot}$$

and if $D^{\cdot}>0$, no matter what $\varepsilon>0$ is taken, we have $n/\mu_n \leqslant (1+\varepsilon)D^{\cdot}$ for sufficiently large n; hence $\rho n/\mu_n(1+\varepsilon)\rho \leqslant \rho D^{\cdot}\, e^{-1} \leqslant e^{-1}$; and since $x \log x$ decreases in $]0, e^{-1}]$:[3]

$$[\rho n/\mu_n(1+\varepsilon)e] \log [\rho n/\mu_n(1+\varepsilon)e] \geqslant (hD^{\cdot}/e) \log (hD^{\cdot}/e).$$

Therefore it is enough to let p tend to infinity to see that for every $\delta>0$:

$$\limsup \Lambda_n/\lambda_n \leqslant L(a; h, D^{\cdot}) + \delta.$$

The statement of proposition I.1.1 follows from this. The case $D^{\cdot}=0$ is easier to derive, starting out from the inequality above for $\log Q_m$.

The next proposition gives another property of a sequence with finite upper density, which is useful in what follows.

THEOREM I.1.2. *Suppose that $D^{\cdot}<\infty$ and set for $r>0$:*

$$\Lambda_0(r) = \prod \left(1 + \frac{r^2}{\lambda_n^2}\right).$$

This product converges uniformly on every compact set in $[0, \infty[$, and

$$\limsup \frac{\log \Lambda_0(r)}{r} = \pi \bar{D}^{\cdot}.$$

The uniform convergence of the product is a consequence of the fact that $\lambda_n > kn$ with $k>0$.

For $r>0$ we have:

$$\log \Lambda_0(r) = \sum \log \left(1 + \frac{r^2}{\lambda_n^2}\right) = \int_0^\infty \log \left(1 + \frac{r^2}{x^2}\right) dN(x),$$

and integration by parts yields

$$\log \Lambda_0(r) = 2r^2 \int_0^\infty \frac{D(x)\, dx}{x^2 + r^2};$$

the terms that are not under the integral sign disappear on integrating by parts, because on the one hand $N(0)=0$, and on the other hand $\lim N(x) \log (1+r^2/x^2)=0$, since $D^{\cdot}<\infty$.

Integrating by parts a second time we obtain:

$$\log \Lambda_0(r) = -2r^2 \int_0^\infty x\bar{D}(x)\, d\left(\frac{1}{x^2+r^2}\right).$$

This enables us to write for $x>0$:

$$\begin{aligned}\log \Lambda_0(r)/r &= -2r\int_0^x y\bar{D}(y)\, d\left(\frac{1}{y^2+r^2}\right)\\ &\quad -2r\int_x^\infty y\bar{D}(y)\, d\left(\frac{1}{y^2+r^2}\right)\\ &\leqslant 2\max_{0\leqslant y\leqslant x}\bar{D}(y)x^3/r(x^2+r^2)+\pi\sup_{y\geqslant x}\bar{D}(y);\end{aligned}$$

and since lim $\sup_{y\geqslant t}\bar{D}(y)=\bar{D}^{\cdot}$, we obtain the required inequality.

I.2. General properties of convergence

Let Λ be a sequence with strictly increasing positive terms tending to infinity, and $A=\{a_n\}$ a sequence of complex numbers.

The series

$$\text{(I.1)}\qquad \sum a_n e^{-\lambda_n s},$$

where $s=\sigma+it$ (σ and t being real), is called a *Dirichlet series*. For given sequences Λ and $A=\{a_n\}$ we denote the series (I.1) by (A, Λ).

If in a Taylor series (with $a_0=0$)

$$\text{(I.2)}\qquad \sum a_n z^n$$

we set $z=e^{-s}$, then the series becomes a Dirichlet series

$$\text{(I.3)}\qquad \sum a_n e^{-ns}$$

which we call a *Taylor-D series*. Here $\lambda_n=n$ $(n\geqslant 1)$.

Another example of a Dirichlet series is

$$\text{(I.4)}\qquad \sum\frac{1}{n^s}=\sum e^{-(\log n)s}$$

which represents the Riemann Zeta-function $\zeta(s)$, a function that plays a principal role in the theory of the distribution of prime numbers.

Let us recall that a Taylor series has a radius of convergence R, which may be zero, infinity, or a finite positive number. In the first case the series converges for $z=0$ only; if $0<R<\infty$, the series converges for $|z|<R$, and does not converge for any value z with $|z|>R$. The disk $|z|<R$ and the circle $|z|=R$ are, respectively, the disk and the circle of convergence of the series.

In any case R is given by the formula

$$R^{-1} = \limsup |a_n|^{1/n}.$$

Moreover, it is known that the series converges uniformly on every compact subset of the disk of convergence; in the disk it converges absolutely.

Since the transformation $z=e^{-s}$ gives $\sigma=-\log|z|$, $t=-\arg z$, we see that when z varies on a curve L in the interior of the disk $|z|<a$, then s varies on a curve $L'=-\log L$ situated in the half-plane $\sigma>-\log a$. (If we associate with a point z_0 of L the point $s_0=\sigma_0+it_0$ with $0\leqslant t_0<2\pi$, then the curve L' is well-defined, for we agree to let s vary continuously when z varies on L.)

The series (I.3) then converges absolutely as long as

$$\text{(I.5)} \qquad \sigma > \limsup \frac{\log|a_n|}{n} = \sigma_c$$

(if $\sigma_c<\infty$), and does not converge for $\sigma<\sigma_c$ (provided that $-\infty<\sigma_c$). The series also converges uniformly on every compact set of the half-plane (I.5). The series (I.3) does not converge anywhere if $\sigma_c=\infty$.

What we have just said concerning the series (I.3) cannot be transferred directly to every Dirichlet series simply by replacing n by λ_n. From the point of view of convergence, the essential difference lies in this: that different half-planes of convergence do not coincide.

We begin with the following theorem:

THEOREM I.2.1. *If* $\sum a_n e^{-\lambda_n s_0}$ $(s_0=\sigma_0+it_0)$ *converges, then the series* (A, Λ) *converges uniformly on every closed angle given by*

$$|\arg(s-s_0)| \leqslant \gamma < \pi/2.$$

We set $s=s_0+s'$, $s'=\sigma'+it'$, $\sigma'>0$, $|\arg s'|<\gamma$, and we write

$$\sum_{n\leqslant m} a_n e^{-\lambda_n s_0} = A_m(s_0) = A_m$$

$$\sum a_n e^{-\lambda_n s_0} = S = \lim A_m.$$

We have

$$\sigma' > 0, \quad |t'|/\sigma' \leqslant \tan\gamma = M < \infty.$$

For $s=s_0+s'$ and $q>p\geqslant 2$ we have:

$$\begin{aligned}\sum_{p\leqslant n\leqslant q} a_n\, e^{-\lambda_n s} &= \sum_{p\leqslant n\leqslant q} a_n\, e^{-\lambda_n s_0}\, e^{-\lambda_n s'}\\ &= \sum_{p\leqslant n\leqslant q} (A_n - A_{n-1})\, e^{-\lambda_n s'}\\ &= \sum_{p\leqslant n\leqslant q} [(A_n - S) - (A_{n-1} - S)]\, e^{-\lambda_n s'}\\ &= \sum_{p\leqslant n\leqslant q} [(A_n - S) - (A_{n-1} - S)]\, e^{-\lambda_n s'}\\ &= \sum_{p\leqslant n\leqslant q-1} (A_n - S)(e^{-\lambda_n s'} - e^{-\lambda_{n+1} s'})\\ &\quad + (A_q - S)\, e^{-\lambda_q s'} - (A_{p-1} - S)\, e^{-\lambda_p s'}.\end{aligned}$$

Given $\varepsilon>0$ we choose p so that $|A_n-S|<\varepsilon$ for $n\geqslant p-1$; then we have:

$$\begin{aligned}\sum_{p\leqslant n\leqslant q} a_n\, e^{-\lambda_n s} &\leqslant \sum_{p\leqslant n\leqslant q-1} |A_n - S|\cdot|e^{-\lambda_n s'} - e^{-\lambda_{n+1} s'}|\\ &\quad + |A_q - S|\, e^{-\lambda_q \sigma'} + |A_{p-1} - S|\, e^{-\lambda_p \sigma'}\\ &\leqslant \varepsilon \sum_{p\leqslant n\leqslant q-1} |e^{-\lambda_n s'} - e^{-\lambda_{n+1} s'}| + 2\varepsilon.\end{aligned} \tag{I.6}$$

Now

$$\begin{aligned}|e^{-\lambda_n s'} - e^{-\lambda_{n+1} s'}| &= |s'| \left| \int_{\lambda_n}^{\lambda_{n+1}} e^{us'}\, du \right| \leqslant |s'| \int_{\lambda_n}^{\lambda_{n+1}} e^{u\sigma'}\, du\\ &= \frac{|s'|}{\sigma'} (e^{-\lambda_n \sigma'} - e^{\lambda_{n+1} \sigma'})\\ &\leqslant (M+1)(e^{-\lambda_n \sigma'} - e^{\lambda_{n+1} \sigma'}),\end{aligned}$$

and it follows from (I.6) that

$$\begin{aligned}\left| \sum_{p\leqslant n\leqslant q} e^{-\lambda_n s} \right| &\leqslant 2\varepsilon + \varepsilon(M+1) \sum_{p\leqslant n\leqslant q-1} (e^{-\lambda_n \sigma'} - e^{\lambda_{n+1} \sigma'})\\ &\leqslant \varepsilon(3+M),\end{aligned}$$

which proves the theorem.

Here is an immediate corollary to the preceding theorem.

THEOREM I.2.2. *If* $\sum a_n\, e^{-\lambda_n s_0}$ $(s_0=\sigma_0+it_0)$ *converges, then the series* $\sum a_n\, e^{-\lambda_n s}$ *converges at every point* $s=\sigma+it$ *with* $\sigma>\sigma_0$, *and converges uniformly on every compact set contained in the half-plane* $\sigma>\sigma_0$.

If we replace the series (A, Λ) by

$$\sum |a_n| e^{-\lambda_n s}, \tag{I.7}$$

Theorem I.2.2 permits us to assert

THEOREM I.2.3. *If the series (A, Λ) converges absolutely for $s = s_0 = \sigma_0 + it_0$, it also converges absolutely for $s = \sigma + it$ with $\sigma > \sigma_0$.*

Consequently,

THEOREM I.2.4. *If (A, Λ) converges at any point of the plane, two cases are possible:* (1) *(A, Λ) converges for every value of s,* (2) *there exists a number σ_c $(-\infty < \sigma_c < \infty)$ such that (A, Λ) converges for every s with $\sigma > \sigma_c$ and does not converge for any s with $\sigma < \sigma_c$.*

If (A, Λ) converges absolutely at any point of the plane, two cases are possible: (1) *(A, Λ) converges absolutely in the whole plane,* (2) *there exists a number σ_a $(-\infty < \sigma_a < \infty)$ such that (A, Λ) converges absolutely for $\sigma > \sigma_a$ and (A, Λ) does not converge absolutely for any point of the half-plane $\sigma < \sigma_a$.*

If (A, Λ) does not converge at any point, we write $\sigma_c = \infty$.

If (A, Λ) does not converge absolutely at any point, we write $\sigma_a = \infty$.

If (A, Λ) converges at every point, we write $\sigma_c = -\infty$, and if it converges absolutely at every point, we write $\sigma_a = -\infty$.

The quantities σ_c and σ_a are called, respectively, the *abscissa of convergence* and *abscissa of absolute convergence* of the series (A, Λ), and the straight lines $\sigma = \sigma_c$ and $\sigma = \sigma_a$ are, respectively, the *axis of convergence* and the *axis of absolute convergence* of (A, Λ). Evidently $\sigma_a \geqslant \sigma_c$. For a Taylor-D series $(\lambda_n = n)$ we have $\sigma_a = \sigma_c$, but this is not true in general. It is known, for example, that the series

$$\sum (-1)^n n^{-s} = \sum (-1)^n e^{-(\log n)s}$$

converges for $\sigma > 0$ and does not converge for $\sigma < 0$. By contrast, the series $\sum n^{-s}$ converges for $\sigma > 1$ and does not converge for $\sigma \leqslant 1$. For the series $\sum (-1)^n n^{-s}$ we have $\sigma_c = 0$, $\sigma_a = 1$.

Furthermore, a Dirichlet series may converge in the entire plane and not converge absolutely anywhere. Here is an example.

We set $a_n = (-1)^n n^{-1}$, $\lambda_n = (\log \log n)^{1/2}$ $(n \geqslant 3)$. For $k > 0$ we have the following relations:

$$\lim a_n e^{\lambda_n k} = 0$$

$$|a_n| e^{\lambda_n k} < |a_{n-1}| e^{\lambda_{n-1} k} \quad (n > n_k).$$

The first relation is evident, the second can be written in the form

$$-\log\left(1-\frac{1}{n}\right) > k\,[(\log\log n)^{1/2} - (\log\log(n-1))^{1/2}] \qquad (n > n_k)$$

(this follows from $\log((n-1)/n) \sim -n^{-1}$ and the relation

$$(\log\log n)^{1/2} - (\log\log(n-1))^{1/2} \sim (2n\log n(\log\log n)^{1/2})^{-1}).$$

Hence the series $\sum a_n\, e^{-\lambda_n k}$ converges for every k, and this proves that $\sigma_c = -\infty$.

On the other hand, it is obvious that $\sum n^{-1}\, e^{-(\log\log n)^{1/2}\, k}$ diverges for every k, hence $\sigma_a = \infty$.

Note that in our two examples

$$(\lambda_n = \log n, \qquad \lambda_n = (\log\log n)^{1/2})$$

we have $D^{\cdot} = \infty$. But the following elementary proposition holds:

THEOREM I.2.5. *If* $D^{\cdot} < \infty$, *then*

$$\sigma_a = \sigma_c = \limsup \frac{\log|a_n|}{\lambda_n}.$$

Suppose that $n/\lambda_n \leqslant L^{-1} < \infty$ $(n \geqslant 1)$ and suppose, to begin with, that $\limsup(\log|a_n|/\lambda_n) = a < \infty$. If $b > a$, we have for $n > n_b$: $|a_n| \leqslant e^{\lambda_n b}$ and for $\sigma > b$:

$$|a_n|\, e^{-\lambda_n \sigma} \leqslant e^{\lambda_n(b-\sigma)} < e^{L(b-\sigma)n}.$$

Consequently $\sum |a_n|\, e^{-\lambda_n \sigma} < \infty$ for $\sigma > a$. On the other hand, if $-\infty < a < \infty$ and $-\infty < \sigma < b < a$, then we have for infinitely many n: $|a_n| > e^{\lambda_n b}$ and $|a_n|\, e^{-\lambda_n \sigma} \geqslant e^{\lambda_n(b-\sigma)} > 1$ $(n > 1)$; hence the series (A, Λ) does not converge for $\sigma < a$. This proves the proposition.

In the proposition we have just proved the condition $D^{\cdot} < \infty$ can be replaced by the rather less restrictive condition: $\lim \log n/\lambda_n = 0$. This result (which is due to Valiron) will not be proved here.

But we prove the following theorem:

THEOREM I.2.6. *For every Dirichlet series*

$$\sigma_a - \sigma_c \leqslant \limsup \frac{\log n}{\lambda_n}.$$

We denote by b the value of the expression on the right. We have to show that for every $\varepsilon > 0$

$$\text{(I.8)} \qquad \sum |a_n| \, e^{-\lambda_n(\sigma_c + b + \varepsilon)} < \infty.$$

From the convergence of (A, Λ) for $s = \sigma_c + \varepsilon/2$ it follows that

$$\sum |a_n| \, e^{-\lambda_n(\sigma_c + \varepsilon/2)} < A < \infty$$

and

$$|a_n| \, e^{-\lambda_n(\sigma_c + b + \varepsilon)} < A \, e^{-\lambda_n(b + \varepsilon/2)};$$

but for $n > n_\varepsilon$: $\log n/\lambda_n \leqslant b + \varepsilon/4$, which proves that the general term of (I.8) for $n > n_\varepsilon$ is smaller than $An^{-(b+\varepsilon/2)/(b+\varepsilon/4)}$, and so the theorem is proved.

Finally, here is a theorem which gives the abscissae of convergence for an arbitrary sequence Λ:

THEOREM I.2.7. *We set*

$$a = \limsup \left(\log \Big| \sum_{m \leqslant n} a_m \Big| / \lambda_n \right).$$

If $a < \infty$, *the series* (A, Λ) *converges for* $\sigma > \max(0, a)$, *and if* $a > 0$, *then* $\sigma_c = a$.

We set

$$b = \limsup \left(\log \sum_{m \leqslant n} |a_m| \right) / \lambda_n.$$

If $b < \infty$, *then the series* (A, Λ) *converges absolutely for* $\sigma > \max(0, b)$, *and if* $b > 0$, *we have* $\sigma_a = b$.

The part of the theorem concerning absolute convergence is an immediate consequence of the first part if in (A, Λ) we replace a_n by $|a_n|$.

We set $A_n = \sum_{m \leqslant n} a_m$. For $\varepsilon > 0$ and $n > n_\varepsilon$ we have

$$\log |A_n| / \lambda_n < a + \varepsilon/2$$

if $a > -\infty$, and $\log |A_n|/\lambda_n < \varepsilon/2$ if $a = -\infty$. Hence

$$\text{(I.9)} \qquad |A_n| < e^{\lambda_n(a + \varepsilon/2)}, \quad \text{or} \quad |A_n| < e^{\lambda_n(\varepsilon/2)} \quad (\text{if} \quad a = -\infty).$$

We set $a' = \max(0, a)$. Then for $q > p \geqslant 2$ we have:

$$\begin{aligned}
\sum_{p \leqslant n \leqslant q} a_n e^{-\lambda_n(a'+\varepsilon)} &= \sum_{p \leqslant n \leqslant q} (A_n - A_{n-1}) e^{-\lambda_n(a'+\varepsilon)} \\
&= \sum_{p \leqslant n \leqslant q-1} A_n(e^{-\lambda_n(a'+\varepsilon)} - e^{-\lambda_{n+1}(a'+\varepsilon)}) \\
&\quad + A_q e^{-\lambda_q(a'+\varepsilon)} - A_{p-1} e^{-\lambda_p(a'+\varepsilon)}.
\end{aligned}$$

From (I.9) it follows that for $p > 1 + n_\varepsilon$

$$\begin{aligned}
\Big| \sum_{p \leqslant n \leqslant q} a_n e^{-\lambda_n(a'+\varepsilon)} \Big| &\leqslant \sum_{p \leqslant n \leqslant q-1} e^{\lambda_n(a'+\varepsilon/2)} \\
&\quad \times (a^{-\lambda_n(a'+\varepsilon)} - e^{-\lambda_{n+1}(a'+\varepsilon)}) \\
&\quad + e^{\lambda_q(a'+\varepsilon/2)} e^{-\lambda_q(a'+\varepsilon)} \\
&\quad + e^{\lambda_{p-1}(a'+\varepsilon/2)} e^{-\lambda_p(a'+\varepsilon)} \\
&\leqslant \sum_{p \leqslant n \leqslant q-1} e^{\lambda_n(a'+\varepsilon/2)}(a'+\varepsilon) \\
&\quad \times \int_{\lambda_n}^{\lambda_{n+1}} e^{-u(a'+\varepsilon)}\, du \\
&\quad + e^{-\lambda_q(\varepsilon/2)} + e^{-\lambda_p(\varepsilon/2)} \\
&\leqslant (a'+\varepsilon) \int_{\lambda_p}^{\lambda_q} e^{-u(\varepsilon/2)}\, du \\
&\quad + e^{-\lambda_q(\varepsilon/2)} + e^{-\lambda_p(\varepsilon/2)}.
\end{aligned}$$

This last expression tends to zero as p tends to infinity. Hence the series (A, Λ) converges for $s = a' + \varepsilon$, consequently $\sigma_c < a'$.

We are now going to show that if (A, Λ) converges for $s = \sigma_0 > 0$, then $a \leqslant \sigma_0$. From this it follows that if $a > 0$, then also $a \leqslant \sigma_c$. This fact, together with the inequality $\sigma_c \leqslant a'$, which we have already proved, yields the theorem.

We can write

$$\begin{aligned}
A_n &= \sum_{m \leqslant n} a_m \\
&= \sum_{2 \leqslant m \leqslant n-1} A_m(\sigma_0)(e^{\lambda_m \sigma_0} - e^{\lambda_{m+1}\sigma_0}) \\
&\quad + A_n(\sigma_0) e^{\lambda_n \sigma_0} + a_1 e^{\lambda_1 \sigma_0}(e^{\lambda_1 \sigma_0} - e^{\lambda_2 \sigma_0}),
\end{aligned}$$

$(A_m(\sigma_0)=\sum_{n\leqslant m} a_n\, e^{-\lambda_n\sigma_0})$. If $\sum a_n\, e^{-\lambda_n\sigma_0}$ converges, then there exists a constant M such that $|A_n(\sigma_0)|<M\ (n\geqslant 1)$, and

$$|A_n| \leqslant M\Big[\sum_{m\leqslant n\leqslant 1} (e^{\lambda_{m+1}\sigma_0} - e^{\lambda_m\sigma_0}) + e^{\lambda_n\sigma_0}\Big]$$

$$= M(2\, e^{\lambda_n\sigma_0} - e^{\lambda_1\sigma_0}) < 2Me^{\lambda_n\sigma_0},$$

which gives

$$a = \limsup \log |A_n|/\lambda_n \leqslant \sigma_0.$$

The following theorem is an immediate corollary to Theorem I.2.7.

THEOREM I.2.8. *If* $\sum a_n\, e^{-\lambda_n\sigma_0}$ *does not converge, then*

$$\sigma_c = \sigma_0 + \limsup \Big(\log \Big|\sum_{n\leqslant m} a_m e^{-\lambda_m\sigma_0}\Big|/\lambda_n\Big).$$

If $\sum |a_n|\, e^{-\lambda_n\sigma_0}=\infty$, *then*

$$\sigma_a = \sigma_0 + \limsup \Big(\log \sum_{n\leqslant m} |a_m|\, e^{-\lambda_m\sigma_0}/\lambda_n\Big).$$

We ought to remark that in Theorem I.2.7 the condition $a>0$ (as well as the condition $b>0$) is essential. In fact, suppose that the series (I.2) represents a function $f(z)$ holomorphic in $|z|<R_1$ with $R_1>1$ and such that $f(1)\neq 0$. The series

$$\varphi(z) = \frac{1}{1-z} f(z) = \sum A_n z^n \Big(A_n = \sum_{n\leqslant m} a_m\Big)$$

has radius of convergence 1, for $z=1$ is a singular point of $\varphi(z)$, and consequently

$$\limsup (\log |A_n|/n) = 0,$$

however, for the series (I.3) we have $\sigma_c \leqslant -\log R_1 < 0$.

I.3. The calculus of coefficients and some important combinations of coefficients

First we are going to extend to Dirichlet series the classical Cauchy formula for the coefficients of a Taylor series. We assume that $\sigma_a<\infty$, although this condition is not essential. But we do not have to use the most general cases when the formulae which we establish here subject to this restrictive condition are still valid.

THEOREM I.3.1. *If* $\sigma_a < \infty$, *and if* $f(s)$ *denotes the sum of* (A, Λ) *for* $\sigma > \sigma_a$, *then*

$$a_n\, e^{-\lambda_n \sigma_1} = \lim \frac{1}{T} \int_{t_0}^{T} f(\sigma_1 + it)\, e^{\lambda_n it}\, dt \quad (n \geqslant 1),$$

where t_0 *is arbitrary and* σ_1 *is an arbitrary real number with* $\sigma_1 < \sigma_a$.

We have for $s = \sigma_1 + it$:

$$\begin{aligned} \frac{1}{T} \int_{t_0}^{T} f(s)\, e^{\lambda_n s}\, dt &= \frac{1}{T} \int_{t_0}^{T} \sum a_m\, e^{(\lambda_n - \lambda_m)s}\, dt \\ &= \frac{1}{T} \int_{t_0}^{T} \Big(\sum_{m \leqslant n-1} a_m\, e^{(\lambda_n - \lambda_m)s} \Big)\, dt + \frac{1}{T} \int_{t_0}^{T} a_n\, dt \\ &\quad + \frac{1}{T} \int_{t_0}^{T} \Big(\sum_{m \geqslant n+1} a_m\, e^{(\lambda_n - \lambda_m)s} \Big)\, dt. \end{aligned}$$

Since for a real number k with $k \neq 0$:

$$\lim \frac{1}{T} \int_{t_0}^{T} e^{k(\sigma_1 + it)}\, dt = 0,$$

and since the series under the integral signs converge uniformly with respect to t, $t \in\,]-\infty, \infty[$, we see that

$$\lim \frac{1}{T} \int_{t_0}^{T} \Big(\sum_{m \geqslant n-1} a_m\, e^{(\lambda_n - \lambda_m)s} \Big)\, dt = 0$$

$$\lim \frac{1}{T} \int_{t_0}^{T} \Big(\sum_{m \geqslant n+1} a_m\, e^{(\lambda_n - \lambda_m)s} \Big)\, dt = 0.$$

Consequently,

$$\lim \frac{1}{T} \int_{t_0}^{T} f(s)\, e^{\lambda_n s}\, dt = \lim \frac{1}{T} \int_{t_0}^{T} a_n\, dt = a_n,$$

which corresponds to the statement of the theorem.

Strictly speaking, we see that in the theorem just proved we can replace σ_a by σ_u, the abscissa of uniform convergence of (A, Λ), that is, the greatest lower bound of the quantities σ' such that (A, Λ) converges uniformly in every half-plane $\sigma \geqslant \sigma' + \varepsilon$, where $\varepsilon > 0$ is arbitrary. For in the proof we have only used the uniform convergence of the series. Now we need the following lemma:

I.3.2. *For $c > 0$, k a positive integer, and ω real, we have:*

$$\frac{1}{2\pi} \int \frac{e^{\omega(c+it)}}{(c+it)^k}\, dt \begin{cases} = \dfrac{\omega^{k-1}}{(k-1)!} & (\omega > 0) \\ = 0 & (\omega \leqslant 0) \end{cases}$$

Suppose first that $\omega > 0$. For $T > 0$, $\sigma_1 < 0$ we denote by $I_1(\pm T, \sigma_1)$ the intervals $(\sigma_1 \leqslant \sigma \leqslant c, t = \pm T)$ and by $I_2(T, \sigma_1)$ and $I_3(T)$ the intervals $(\sigma = \sigma_1, |t| \leqslant T)$ and $(\sigma = c, |t| \leqslant T)$. Let $R(T, \sigma_1)$ be the rectangle formed by these four intervals. By the residue theorem we have

$$\frac{1}{2\pi i} \oint_{R(T,\sigma_1)} \frac{e^{\omega s}}{s^k}\, ds = \frac{\omega^{k-1}}{(k-1)!}.$$

But since for fixed σ_1 the integrals extended over $I_1(T, \sigma_1)$ and $I_1(-T, \sigma_1)$ tend to zero, we see that

$$\frac{1}{2\pi} \int \frac{e^{\omega(c+it)}}{(c+it)^k}\, dt - \frac{1}{2\pi} \int \frac{e^{\omega(\sigma_1+it)}}{(\sigma_1+it)^k}\, dt = \frac{\omega^{k-1}}{(k-1)!}.$$

And since the second integral in this equality tends to zero as σ_1 tends to $-\infty$, the lemma is proved for $\omega > 0$. For $\omega < 0$ the proof is similar. This time $R(T, \sigma_1)$ must be replaced by the contour $R'(T, \sigma_2)$ with $\sigma_2 > c$, composed of the intervals:

$$I_1'(\pm T, \sigma_2) \equiv (c \leqslant \sigma \leqslant \sigma_2, t = \pm T), I_2(T, \sigma_2)$$
$$\equiv (\sigma = \sigma_2, |t| \leqslant T), I_3(T).$$

The integral written above, but this time extended over the contour $R'(T, \sigma_2)$ (instead of $R(T, \sigma_1)$) is zero, because the integrand is holomorphic in the interior and on $R'(T, \sigma_1)$, and if we first let T tend to $+\infty$ and then σ_2 to $+\infty$, we obtain the required result.

The following theorem will be very useful later on.

THEOREM I.3.3. *Suppose that* $\sigma_a < \infty$. *If* $\nu > 0$ *and if* k *is a positive integer, we have*

$$\frac{(k-1)!}{2\pi}\int \frac{f(c+it)\,e^{\nu(c+it)}}{(c+it)^k}\,dt \begin{cases} = \sum_{\lambda_n < \nu} (\nu - \lambda_n)^{k-1} a_n, & \text{for } \nu > \lambda_1 \\ = 0 & \text{for } \nu \leqslant \lambda_1, \end{cases}$$

where $c > \max(\sigma_a, 0)$ *and* $f(s)$ *is the sum of* (A, Λ).

For $\nu > \lambda_1$ and $s = c + it$ we have

$$\int \frac{f(s)\,e^{\nu s}}{s^k}\,dt = \int \frac{\sum_{\lambda_n < \nu} a_n\,e^{(\nu-\lambda_n)s}}{s^k}\,dt + \int \frac{\sum_{\lambda_n \geqslant \nu} a_n\,e^{(\nu-\lambda_n)s}}{s^k}\,dt.$$

From I.3.2 and the uniform convergence of the series under the last integral sign we see that this integral is zero. The same Lemma I.3.2. allows us to write:

$$\frac{(k-1)!}{2\pi}\int \frac{f(s)}{s^k}\,e^{\nu s}\,dt = \frac{(k-1)!}{2\pi}\sum_{\lambda_n < \nu} a_n \int \frac{e^{(\nu-\lambda_n)s}}{s^k}\,dt$$
$$= \sum_{\lambda_n < \nu} a_n(\nu - \lambda_n)^{k-1}.$$

If $\nu \leqslant \lambda_1$, the integral is zero. In this theorem we can likewise replace σ_a by σ_u.

We end this chapter with an elementary inequality, which generalizes the Cauchy inequality for the Taylor coefficients.

THEOREM I.3.4. *Let* $\sigma_1 > \sigma_a$, t_0 *be any real number and* $M = \sup_{t \geqslant t_0} |f(\sigma_1 + it)|$, *where* $f(s)$ *is the sum of* (A, Λ). *Then for* $n \geqslant 1$:

(I.10) $\quad |a_n| \leqslant M\,e^{\lambda_n \sigma_1}.$

This follows immediately from I.3.1.

NOTES

[1] Unless the contrary is stated, the first index of a sequence $\Delta = \{\lambda_n\}$ is always 1 and not 0; when the index or the variable in lim sup, lim inf, or lim tends to $+\infty$, this is not stated explicitly. Also $\limsup_{n\to\infty} p_n$ is written $\limsup p_n$.

[2] $E[a]$ denotes the integral part of a.

[3] We recall that $hD^{\cdot} \leqslant 1$.

CHAPTER II

INEQUALITIES CONCERNING THE COEFFICIENTS

II.1. Inequalities corresponding to the arithmetic character of the exponents

Suppose that $\sigma_a \leqslant 0$, and let $T(\sigma)$ be an arbitrary function with $T(\sigma) > -\infty$ for $\sigma > \sigma_a$. We write $\mathfrak{D}_T$ for the set of points $s = \sigma + it$ with $\sigma > \sigma_a$, $t > T(\sigma)$. If $f(s)$ is the sum of (A, Λ) for $\sigma > \sigma_a$, we set

$$M = \sup_{s \in \mathfrak{D}_T} |f(s)|;$$

obviously

$$|a_n| \leqslant M,$$

for (I.10) holds by I.3.4, for every $\sigma_1 > 0$.

We are now going to show that if a member λ_k of the sequence Λ 'is distinguished' by an arithmetic property from the other members of the sequence – this distinguishing feature could express itself, for example, in the form of the difference $\lambda_k - \lambda_m$ for all positive integers m with $m \neq k$ – then it suffices that the analytic continuation $f(s)$ of (A, Λ) should be bounded by M in a domain $\mathfrak{D}$ *whose properties depend on the arithmetical character of* λ_k, to be able to state that $|a_k| \leqslant BM$, where B is a numerical constant independent of Λ and of k. This constant depends only on the arithmetical character of k. No inequality of this form is to be valid if we abandon this assumption that λ_k is 'distinguished'. (In other words, the inequality becomes false no matter what value is chosen for B.) If λ_k is an integer, then the value of $\min(\lambda_k - \lambda_{k-1}, \lambda_{k+1} - \lambda_k)$ plays the same role as the arithmetical character of the difference $\lambda_k - \lambda_m$ $(m \neq k)$.

We begin with some definitions.

In the plane of the complex variable $z = x + iy = re^{i\theta}$ $(r = |z|)$ we use the name *starred curve*, which we denote by a capital letter with an asterisk, for example Γ^*, for any closed Jordan curve that is symmetric with respect to the real axis, passes through the point $z = 1$, and is given by a continuous function $r = \varphi(\theta)$ $(\theta \in [0, 2\pi]$, $\varphi(0) = \varphi(2\pi))$ $\varphi(\theta) > 0$, where φ has a continuous derivative in the neighbourhood of $\theta=0$, subject to the condition

$$\lim_{\theta\to 0}\frac{dr}{d\theta}\cdot\frac{1}{\theta}=c,$$

and where c is a finite quantity.

We denote by $D(\Gamma^*)$ the domain bounded by Γ^*.

We say that a starred curve G^* '*shelters*' *the starred curve* Γ^* if for every $z\in\Gamma^*$, $z\neq 1$, we have $z\in D(G^*)$ and if by writing for $z\in G^*$

$$\lim_{\theta\to 0}\frac{dr}{d\theta}\cdot\frac{1}{\theta}=c_1,$$

we have $c_1>c$.

Let $\Phi(z)$ be a function holomorphic on the closed disk $|z|\leqslant 1$, real for real z, with $\Phi(1)=1$, $\Phi'(1)>-1$, $\Phi(z)\neq 0$ for $|z|<1$. Such a function is said to be *attached to a starred curve* Γ^* if for $|z|<1$ we have $z\Phi(z)\in D(\Gamma^*)$. Let $\lambda>0$; an integer $m>0$ is called λ-*suitable* for the function Φ if by writing

$$\text{(II.1)}\qquad (\Phi(z))^\lambda = d_0^{(\lambda)}+\sum d_m^{(\lambda)}z^m,$$

we have $d_m^{(\lambda)}\neq 0$. Two numbers $\lambda>0$ and $\lambda'>0$ are said to be *distinct modulo 1 relative to* Φ if for no pair $m\geqslant 0$, $m'\geqslant 0$ of λ- and λ'-suitable numbers for Φ, respectively, we can have the relation

$$\lambda-\lambda'=m'-m.$$

If Δ is any (open) domain containing the origin, we write $\mathscr{L}\Delta$ for the domain of the plane of $s=\sigma+it$ that consists of all the points for which $e^{-s}\in\Delta$. Thus, $\mathscr{L}D(\Gamma^*)$ is the set of all points of the s-plane such that the points $z=e^{-s}$ form a domain whose boundary is Γ^*.

We denote the half-plane $\sigma>\sigma'$ by $P_{\sigma'}$.

Let $\mathfrak{D}$ be a domain of the s-plane containing a half-plane $P_{\sigma'}$ whose boundary is a curve defined by a continuous function $\sigma=\sigma(t)$ ($t\in\,]-\infty,\infty[$, $s=\sigma+it$).

If $\mathscr{F}(s)$ is a function holomorphic in $P_{\sigma'}$, we say that it has *a direct analytic continuation into* $\mathfrak{D}$ *starting from* $P_{\sigma'}$ if $CP_{\sigma'}\cap\mathfrak{D}\neq\emptyset$ and if there exists a holomorphic function in $\mathfrak{D}$ that is equal to $\mathscr{F}(s)$ in $P_{\sigma'}$. We denote this analytic continuation into $\mathfrak{D}$ (and only the direct one) by the same letter $\mathscr{F}$.

We can now state the following general theorem; its character will be easier to grasp by the special cases which we state below.

THEOREM II.1.1. *Let* G^* *be a starred curve,* Γ^* *a starred curve sheltered by* G^* *and* Φ *the function attached to* Γ^*.

If (A, Λ) *has an abscissa of absolute convergence* $\sigma_a < \infty$ *and if its sum* $f(s)$ *has a direct analytic continuation starting from* $P_{\sigma'}$, *where* $\sigma' > \sigma_a$, *into* $\mathscr{L}D(G^*)$, *then there exists a constant* H *depending only on* G^* *and on* Φ, *such that for every* $\lambda_k \in \Lambda$ *that is distinct modulo 1 relative to* Φ *from all the other* λ_n *we have*

$$\text{(II.2)} \qquad |a_k| \leqslant H \sup_{s \in \mathscr{L}D(G^*)} |f(s)| \inf \left(|d_m^{(\lambda_k)}| \sqrt{\lambda_k + m}\right)^{-1}.$$

Here the $d_m^{(\lambda)}$ are defined in (II.1).

A particular choice of a curve Γ^* on one hand and the function Φ attached to it on the other (a choice that permits us to make the properties demanded of the 'distinguished' exponent λ_k more precise) leads from Theorem II.1.1 to the following theorems, which can be regarded as particular cases.

THEOREM II.1.2. *To every starred curve* G^* *sheltering* $|z| = 1$ *there corresponds a constant* B *depending only on* G^* *such that if* $\sigma_a < \infty$ *and if the function* $f(s)$ *defined by* (A, Λ) *has a direct analytic continuation starting from* $P_{\sigma'}$, *where* $\sigma' \geqslant \sigma_a$, *into* $\mathscr{L}D(G^*)$ *the following inequality holds for every* k:

$$\text{(II.3)} \qquad |a_k| \leqslant B \sup_{s \in \mathscr{L}D(G^*)} |f(s)| \lambda_k^{-1/2}.$$

THEOREM II.1.3. *Suppose that a starred curve* G^* *shelters the (starred) curve defined by* $r = \cos(\theta/3)$ $(|\theta| \leqslant \pi)$. *Suppose also that* $\sigma_a < \infty$ *and that* $f(s)$ *defined by* (A, Λ) *has a direct analytic continuation starting from* $P_{\sigma'}$, *where* $\sigma' \geqslant \sigma_a$, *into* $\mathscr{L}D(G^*)$.

Then there exists a constant B_1 *depending only on* G^* *such that the inequality*

$$\text{(II.4)} \qquad |a_k| \leqslant B_1 \sup_{s \in \mathscr{L}D(G^*)} |f(s)|$$

is valid for every λ_k *for which* $\lambda_k - \lambda_n$ *is not an integer for any* $n \neq k$.

THEOREM II.1.4. *If* λ_k *is an integer and if in the statement of Theorem*

II.1.3 *the condition concerning* λ_k *is replaced by the following: every integral* λ_n $(n \neq k)$ *satisfies one of the following two conditions* $\lambda_n > 2\lambda_k$, $\lambda_k > 2\lambda_n$, *while the other conditions of Theorem* II.1.3 *are preserved, then the inequality* (II.4) *also holds.*

It is clear that if in II.1.4 all the λ_n are integers, we can write $F(z) = \sum a_n z^{\lambda_n}$, and with the assumption concerning λ_k on the one hand and the direct analytic continuation of $F(z)$ into $D(G^*)$ on the other hand, we can put the conclusion into the form

$$|a_k| \leqslant B_1 \sup_{z \in D(G^*)} |F(z)|.$$

THEOREM II.1.5. *The conclusions of Theorems* II.1.3 *and* II.1.4 *remain valid if we replace* G^* *by a starred curve* G_1^* *which shelters the starred curve* Γ^* *consisting of the arc* $r = \cos(\theta/2)$, $|\theta| \leqslant \pi/2$, *and of its symmetric image in the y-axis; here the condition bearing on* λ_k *in Theorem* II.1.3 *is to be replaced by the condition that* $\lambda_k - \lambda_n$ *for* $n \neq k$ *is not an even integer; the condition in Theorem* II.1.4 *bearing on* λ_k *has to be replaced by the following condition: if* λ_k *is an integer, then every integral* λ_n $(n \neq k)$ *for which* $\lambda_n - \lambda_k$, *is even satisfies one of the two conditions* $\lambda_n > 3\lambda_k$, $\lambda_k > 3\lambda_n$.

To establish Theorem II.1.1 we have, first of all, to prove the following lemma:

II.1.6. *Let* Γ^*, G^*, *and* Φ *be defined as in Theorem* II.1.1, *let* C_α $(\alpha \geqslant 0)$ *be the curve in the plane of* $z = re^{i\theta}$ *defined by* $r = e^{\alpha\theta^2}$, $|\theta| \leqslant \pi$. *For sufficiently small* α *the image of* C_α *under* $z\Phi(z)$ *is situated on* $\overline{D(G^*)}$.

(According to the notation introduced above, $\overline{D(G^*)}$ denotes the compact domain bounded by G^*.)

We set $\Phi_1(z) = z\Phi(z)$. Since $\Phi_1(1) = 1$, $\Phi_1'(1) = 1 + \Phi'(1) > 0$, there exists a disk with centre at $z = 1$ in which $\Phi_1(z)$ is a single-valued function; let U_1 be such a disk. There also exists a disk with centre at $\zeta = 1$ in which the inverse function $\Phi_1^{-1}(\zeta)$ of Φ_1 $(\Phi^{-1}(\Phi(z)) = z)$ is single-valued; let U_2 be such a disk. We set $\zeta = \xi + i\eta = \rho e^{i\gamma}$ and denote by Q_α the image of C_α under Φ_1. With every $\alpha > 0$ we can associate an $\varepsilon = \varepsilon(\alpha) > 0$ and a $\gamma = \gamma(\alpha) > 0$, sufficiently small for the following conditions to hold: on the one hand, the part of Q_α which is the image (under Φ_1) of the arc of C_α corresponding to $0 \leqslant \theta \leqslant \varepsilon$, is a simple Jordan arc $\mathscr{J}$ which links the point $\zeta = 1$ to a point in the half-plane $\eta > 0 (\zeta = \xi + i\eta)$, without passing through the real axis;

on the other hand, the ray from the origin ($\xi=0$) making the angle γ with the axis $\eta=0$ cuts $\mathscr{J}$ in a single point p. The curve composed of the part of this ray between the origin and p, the arc $\mathscr{J}$ and the interval $0\leqslant\xi\leqslant 1$ on the real axis then determines a simply-connected domain D. In addition, we can choose $\varepsilon(\alpha)$ and $\gamma(\alpha)$ so that $\varepsilon(\alpha)=\varepsilon(0)$ and $\gamma(\alpha)=\gamma(0)$ (this is possible – it suffices to take $\gamma(\alpha)$ sufficiently small) and such that the domain between the arcs Q_0 and Q_α beginning at $\xi=1$ and ending on the ray (through the origin) making with $\eta=0$ the angle $\gamma(0)=\gamma(\alpha)$ should be contained in U_2. We can also choose $\varepsilon(\alpha)=\varepsilon(0)$ small enough for the corresponding arcs of C_0 (which is an arc of $|z|=1$) and of C_α to be contained in U_1. For a given α, having made the choice of the corresponding quantities ε and γ we refer to the resulting domain D as the 'domain D_α'. The image of D_α under Φ_1^{-1} is denoted by D_α^{-1}. The boundary of D_α^{-1} is evidently a simple closed Jordan curve.

If α is sufficiently small, we can choose the domain D_α in such a way that when we write R_α for the part of Q_α that is part of the boundary of D_α we have $R_\alpha\subset\overline{D(G^*)}$.

This comes from the following arguments. If L is a curve in the ζ-plane passing through $\zeta=1$ that can be represented in a neighbourhood of this point by $\rho=\rho(\gamma)$ (polar coordinates: a single ρ for sufficiently small $|\gamma|$) and such that the (finite) limit

$$\lim_{\gamma\to 0}\frac{d\rho}{d\gamma}\cdot\frac{1}{\gamma}$$

exists, we denote this limit by $c(L)$. The fact that $R_\alpha\subset\overline{D(G^*)}$ for sufficiently small α is due to the following inequality, which is valid for sufficiently small α:

$$c(R_\alpha) < c(G^*).$$

For if we set

$$\frac{d\log\Phi_1(z)}{dz} = A(z) + iB(z),$$

then we have:

$$\lim_{\theta\to 0} A(e^{\alpha\theta^2+i\theta}) = 1 + \Phi'(1) = A > 0$$

$$\lim_{\theta\to 0}\frac{dB(e^{\alpha\theta^2+i\theta})}{d\theta} = \frac{\partial B}{\partial y}\bigg]_{z=1} = \frac{\partial A}{\partial x}\bigg]_{z=1} = A'(1),$$

where $A(x)$ is the restriction of $A(z)$ to the real axis. Writing $A'(1)=A_1$ we easily see that

$$c(\Gamma^*) = -\frac{A_1}{A^2} - \frac{1}{A}, \qquad c(R_\alpha) = \frac{2\alpha}{A} - \frac{A_1}{A^2} - \frac{1}{A},$$

and by definition: $c(G^*) > c(\Gamma^*)$. Thus, it is sufficient that $\alpha > 0$ should be small enough for us to have $c(R_\alpha) < c(G^*)$. And so with a suitable choice of α we have $\Phi_1(D_\alpha) \subset D(G^*)$. If α is sufficiently small, the image under Φ_1 of that part of $D(C_\alpha)$ which is in the exterior of D_α and its image in the real axis is certainly contained in $D(G^*)$. But the same holds for the image under Φ_1 of the part of $D(C_\alpha)$ that is contained in D_α (and its image in the real axis). Therefore, there exists an $\alpha > 0$ such that the image of $D(C_\alpha)$ under Φ_1 is in $D(G^*)$. And this demonstrates the lemma.

Now we proceed to the proof of Theorem II.1.1. According to the conditions of the theorem, the function $\Phi(z)$ does not vanish in the disk $|z| < 1$ (see the definition of a function attached to a curve Γ^*); it is holomorphic on $|z| \leqslant 1$. It may have a finite number of zeros on $|z| = 1$, say, $z_1, \ldots, z_k$.

We choose $\alpha > 0$ sufficiently small so that:

(1) $\Phi_1[D(C_\alpha)] \subset D(G^*)$ $(\Phi_1(z) = z\Phi(z))$, which is possible by virtue of the lemma we have just proved;

(2) $\Phi(z)$ has no zeros on the disk $|z| \leqslant e^{\alpha\pi^2}$ other than the points z_j $(j = 1, 2, \ldots, k)$.

We write L_j $(j = 1, 2, \ldots, k)$ for the ray through the origin passing through the point z_j, and S_j for the segment of L_j between z_j and C_α. We denote by $\mathscr{C}_\alpha$ the curve composed of C_α and of the k segments S_j. Finally, we denote by Δ_α the compact set bounded by $\mathscr{C}_\alpha$.

We set

$$\Psi(s) = f[-\log \Phi_1(e^{-s})],$$

where $\log \Phi_1(e^{-\sigma})$ is real for $\sigma > 0$. The function $\Psi(s)$ is holomorphic on $\mathscr{L}\Delta_\alpha$, and we have

$$\sup_{s \in \mathscr{L}\Delta_\alpha} |\Psi(s)| \leqslant \sup_{s \in \mathscr{L}D(G^*)} |f(s)| = M.$$

In fact, we have $\Delta_\alpha \subset D(G^*)$, and $\Phi_1(z)$ has no zeros in Δ_α other than a simple zero at the origin.

We set

$$\Phi(z) = d_0 + \sum d_m z^m$$

(that is, $d_m = d_m^{(1)}$ according to (II.1)).

We see that for sufficiently small $r > 0$

$$\sum |a_n| (r \sum |d_m| r^m)^{\lambda_n} < \infty,$$

which enables us to write for sufficiently large σ:

$$\begin{aligned}\Psi(s) &= \sum a_n\, e^{-\lambda_n s} (\sum d_m\, e^{-ms})^{\lambda_n} \\ &= \sum a_n \sum d_m^{(\lambda_n)}\, e^{-(m+\lambda_n)s} = \sum l_p\, e^{-\mu_p s},\end{aligned}$$

where the μ_p are of the form $m+\lambda_n$, l_p being the sum extended over all the terms $a_n d_m^{(\lambda_n)}$ for which $\mu_p = m+\lambda_n$.

For every $p \geqslant 1$ and sufficiently large σ_1 we can write:

$$l_p = \lim \frac{1}{T} \int_0^T \Psi(\sigma_1 + it)\, e^{\mu_p(\sigma_1 + it)}\, dt.$$

Now let p be such that $\alpha\pi^2\mu_p \geqslant 1$, and let $C_{\alpha,p}$ denote the curve composed of the arc $\sigma = \mu_p^{-1} - \alpha t^2$, $0 \leqslant t \leqslant \pi$, and the segments $S_{j,p}$, which are the parts of the segments $-\log S_j$ ($s \in -\log S_j \equiv e^{-s} \in S_j$) contained between this arc and the line $\sigma = 0$ (these segments are evidently parallel to the real axis $t = 0$), where the ordinates of these segments belong to $]0, \pi]$. In other words, the segments $S_{j,p}$ which we consider are all of the form

$$t = t_j, \qquad \mu_p^{-1} - \alpha t_j^2 \leqslant \sigma \leqslant 0, \qquad 0 < t_j \leqslant \pi$$
$$\text{with} \quad \Phi(e^{-it_j}) = 0.$$

If we denote by $C_{\alpha,p,m}$ the union of m curves of which one is the union $C'_{\alpha,p}$ of $C_{\alpha,p}$ with its symmetric image in the line $t = \pi$, while the $m-1$ others are those that we obtain from $C'_{\alpha,p}$ by the successive translations $s + 2n\pi i$ ($n = 1, 2, \ldots, m-1$), we see by an evident application of Cauchy's theorem that

$$l_p = \lim \frac{1}{2m\pi i} \int_{C_{\alpha,p,m}} \Psi(s)\, e^{\mu_p s}\, ds,$$

where the integral is taken (for every m) along the segments of $C_{\alpha,p,m}$ that are parallel to the real axis in both senses. (Incidentally, $\Psi(s)$ does not have the same value when we approach such a segment with increasing or with decreasing t, because $\Phi(e^{-it_j}) = 0$.)

Since $|\Psi(s)| \leqslant M$ on $C_{\alpha,p,m}$, we can write:

$$|l_p| \leqslant \frac{M}{\pi}\left[\int_0^\pi e^{\mu_p(\mu_p-1-\alpha t^2)}\,dt + 2\sum_i \int_0^{\alpha t_j^2-\mu_p} e^{-\mu_p\sigma}\,d\sigma\right].$$

It follows that

$$|l_p| \leqslant \frac{M}{\pi}\left[\int_0^\pi e^{1-\alpha\mu_p t^2}\,dt + k\int_0^{\alpha\pi^2-\mu_p^{-1}} e^{-\mu_p\sigma}\,d\sigma\right].$$

But we can write for $0<\alpha<1$ (and $\alpha\pi^2\mu_p \geqslant 1$):

$$\begin{aligned}\int_0^\pi e^{-\alpha\mu_p t^2}\,dt &= \int_0^{\mu_p^{-1/2}} e^{-\alpha\mu_p t^2} + \int_{\mu_p^{-1/2}}^\pi e^{-\alpha\mu_p t^2}\,dt\\ &< \frac{1}{\mu_p^{1/2}} + \mu_p^{1/2}\int_{\mu_p^{-1/2}}^\pi e^{-\alpha\mu_p t^2}\,t\,dt\\ &= \frac{1}{\mu_p^{1/2}} + \frac{\mu_p^{1/2}}{2}\int_{\mu_p^{-1}}^{\pi^2} e^{-\alpha\mu_p y}\,dy\\ &= \frac{1}{\mu_p^{1/2}} + \frac{1}{2\alpha\mu_p^{1/2}}(e^\alpha - e^{-\alpha\pi^2\mu_p}) \leqslant \frac{A}{\mu_p^{1/2}},\end{aligned}$$

where A is a finite constant. Starting from the obvious inequality $|l_p| \leqslant M$ for $\alpha\pi^2\mu_p \leqslant 1$ we also have:

$$|l_p| \leqslant M \leqslant \frac{M}{\pi\alpha^{1/2}\mu_p^{1/2}}.$$

All in all, no matter what p is, we have

$$|l_p| \leqslant K_\alpha M\mu_p^{-1/2}.$$

But, if λ_k is distinct modulo 1 relative to Φ from all the other λ_n $(n\neq k)$, and if $\mu_p=\lambda_k+m$ with $d_m^{(\lambda_k)}\neq 0$, there exists no other couple λ_n, m' $(n\neq k)$ such that $\mu_p=\lambda_n+m'$, $d_{m'}^{(\lambda_n)}\neq 0$. Consequently, according to the definition of μ_p and l_p we have:

$$\mu_p = \lambda_k + m, \qquad l_p = a_k d_m^{(\lambda_k)},$$

and the inequality we have demonstrated for l_p gives

$$|a_k|\,|d_m^{(\lambda_k)}| \leqslant K_\alpha M(\lambda_k + m)^{-1/2}.$$

The conclusion of the theorem is now an immediate consequence.

Next we show that Theorem II.1.2 is a particular case of Theorem II.1.1. The starred curve Γ^* of Theorem II.1.2 is the circle $|z|=1$, the function $\Phi(z)$ attached to this curve Γ^* is the constant 1; and no matter what $\lambda > 0$ is, only the integer $m=0$ is λ-suitable for Φ. Hence λ_n and λ_m are distinct modulo 1 for all m and n relative to Φ (that is, they are simply distinct, Λ being a strictly increasing sequence). So the inequality (II.2), with the conditions of Theorem II.1.2, becomes (II.3).

Now we make the passage from Theorem II.1.1 to Theorems II.1.3 and II.1.4. The (starred) curve in these statements which is sheltered by G^* is the curve Γ^* corresponding to the statement of Theorem II.1.1; it is the exterior contour of the image of $|z|=1$ under $z\Phi(z)=z(1+z)/2$. As long as $\lambda > 0$ is not an integer, every integer $m \geqslant 0$ is λ-suitable for Φ. And if λ is an integer, then m is λ-suitable for Φ whenever $0 \leqslant m \leqslant \lambda$. The inequality (II.1) can also be written:

$$\begin{aligned} |a_k| &\leqslant A \sup_{s \in \mathscr{L}D(G^*)} |f(s)| \inf_{m} \left(|d_m^{(\lambda_k)}| \sqrt{\lambda_k + m}\right)^{-1} \\ &\leqslant A \sup_{s \in \mathscr{L}D(G^*)} |f(s)| \left(|d_{E(\lambda_k/2)}| \sqrt{\lambda_k + E(\lambda_k/2)}\right)^{-1} \end{aligned}$$

where, as before, $E(a)$ denotes the integral part of a. A simple estimate of $d^{(\lambda)}_{E(\lambda/2)}$ yields the required result.

In Theorem II.1.5 the starred curve Γ^* is the exterior contour of the image of the circle $|z|=1$ under $z\Phi(z)=z(1+z^2)/2$. The passage from Theorem II.1.1 to Theorem II.1.5 starts from this point and follows the arguments used in the passage from Theorem II.1.1 to Theorems II.1.3 and II.1.4.

We remark that the conclusion of Theorem II.1.1 is not valid if no condition is imposed on the nature of λ_k (for a given k). Thus, the function $f(s)=\sum(-1)^n n e^{-ns}$ is bounded in a domain $\mathscr{L}D(G^*)$, where G^* corresponds to the statement II.1.3, but $|a_k|=k$ is not bounded independently of k. In fact, one can even take for a_k the quantity $(-1)^k g(k)$, where $g(z)$ is an entire function of minimum type (of exponential type zero), $g(z)$ having positive Taylor coefficients; for $g(n)=|a_n|$ tends to infinity more rapidly than any power of n, and yet one knows that the function $F(z)=\sum g(n)z^n$ only has $z=1$ as singular point, that is, the function

$$f(s) = \sum (-1)^n g(n)\, e^{-ns}$$

is bounded in an $\mathscr{L}D(G^*)$, where G^* corresponds to the statement II.1.3.

II.2. A general inequality for the coefficients corresponding to finite upper density of the exponents

In what follows we assume that the upper density $D^{\cdot}$ of the sequence Λ is finite. From I.2.5 we know that $\sigma_a=\sigma_c=\limsup(\log|a_n|/\lambda_n)$. We also assume that the last quantity is not $+\infty$, so that $\sigma_a=\sigma_c<\infty$. As before, we denote by $f(s)$ the sum of (A, Λ) for $\sigma>\sigma_c$.

We denote by $C(s_0, R)$ the disk with centre at s_0 and radius R. Let L be a Jordan arc given by $s=s(u)$, $u\in[\alpha, \beta]$. The union

$$\mathfrak{C}(L; R) = \bigcup_{s\in L} C(s, R)$$

is called a *channel of width* $2R$ and with L as *central line*. The *extreme disks* of the channel are $C(\alpha, R)$ and $C(\beta, R)$. Throughout we assume that the extreme disk $C(\beta, R)$ lies in P_c. We say that (A, Λ) has a direct analytical continuation into $\mathfrak{C}(L; R)$ if there exists a holomorphic function in $\mathfrak{C}(L; R)$ that is equal to (A, Λ) in $C(\beta, R)$. We frequently omit the adjective 'direct' when indicating this continuation. We also use the letter f, or frequently (A, Λ), to denote the continuation.

We now establish the following general theorem, which plays a very important role in what follows:

THEOREM II.2.1. *Let* $D^{\cdot}<\infty$ *and suppose that* $\sigma_c<\infty$. *If* $f(s)$, *the sum of* (A, Λ), *has an analytic continuation in a channel* $\mathfrak{C}(L; \pi R)$ *with* $R>\bar{D}^{\cdot}$, *such that* $C(s_0, \pi R)$ $(s_0=\sigma_0+it_0)$ *is one of the disks:*

$$C(s_0, \pi R)\subset\mathfrak{C}(L; \pi R),$$

then we have

$$|a_n| \leqslant C(R)\Lambda_n M(s_0)\, e^{\lambda_n\sigma_0},$$

where $C(R)<\infty$ *only depends*^{1} *on* R, *where* $\{\Lambda_n\}$ *is the sequence associated with* Λ, *and where*

$$M(s_0) = \sup_{s\in C(s_0,\pi R)} |f(s)|.$$

We recall that $\bar{D}^{\cdot}$ is the mean upper density of Λ. We set

$$\Lambda_n(z) = \prod_{m\neq n}\left(1-\frac{z^2}{\lambda_m^2}\right) \quad (n\geqslant 1).$$

Since $D^{\cdot} < \infty$, we see immediately that this product converges uniformly on every compact set, so that $\Lambda_n(z)$ is an entire function whose zeros are only the points $\pm\lambda_m$ $(m \neq n)$, and by I.1.2 we have

$$\text{(II.5)} \qquad \limsup \frac{\log \Lambda_n(ir)}{r} \leqslant \pi \bar{D}^{\cdot},$$

because it is evident that the upper density and the mean upper density of the sequence obtained by removing from Λ a single term (or a finite number of terms) are the same as for Λ itself.

It is clear that

$$\max_{|z|=r} |\Lambda_n(z)| = \Lambda_n(ir).$$

Hence the $\Lambda_n(z)$ is an entire function of exponential type not exceeding $\pi\bar{D}^{\cdot}$.

We now set

$$\Lambda_n(z) = \sum \alpha_p^{(n)} z^{2p};$$

we see that $(-1)^p \alpha_p^{(n)} > 0$ with $\alpha_0^{(n)} = 1$.

We introduce the operator $\mathfrak{F}_n(f)$ defined by

$$\text{(II.6)} \qquad \mathfrak{F}_n(f) = \sum \alpha_p^{(n)} f^{(2p)}.$$

This operator represents a holomorphic function in the channel $\mathfrak{C}(L, \pi(R - \bar{D}^{\cdot}))$, because for every $\varepsilon > 0$ with $\varepsilon < R - \bar{D}^{\cdot}$ the series (II.6) converges uniformly (with respect to s) on every closed disk $\overline{C(s, \pi r)}$ with $s \in L$, $0 < r \leqslant R - \bar{D}^{\cdot} - \varepsilon$. It is easy to see first of all that we may argue under the assumption that f is holomorphic on $\overline{\mathfrak{C}(L, \pi R)}$ (the closed channel). For every point s' for which $|s' - s| = \pi(R - \bar{D}^{\cdot} - \varepsilon)$, where $s \in L$, we can write (for every positive integer q):

$$f^{(q)}(s') = \frac{q!}{2\pi i} \oint \frac{f(\zeta)\, d\zeta}{(\zeta - s')^{q+1}}$$

the integral being taken over the circle with centre at s' and of radius $\pi(\bar{D}^{\cdot} + \varepsilon)$, which gives

$$\text{(II.7)} \qquad |f^{(q)}(s')| \leqslant \frac{q!\,M(s)}{(\pi(\bar{D}^{\cdot} + \varepsilon))^q}$$

where $M(s)$ is the maximum of $|f|$ on $\overline{C(s, \pi R)}$.

Since the type of $\Lambda_n(z)$ does not exceed $\pi\bar{D}^{\cdot}$, the series $\sum a_p^{(n)}(2p)!\, z^{-2p}$ has radius of convergence $R \leqslant \pi\bar{D}$ (it converges at the exterior of the circle). Hence the series (II.6) converges by (II.7) uniformly on the disk $C(s, \pi(R-\bar{D}^{\cdot}-\varepsilon))$. The function $F_n(s)=\mathfrak{F}_n[f(s)]$ is holomorphic in the channel $\mathfrak{C}(L, \pi(R-\bar{D}^{\cdot}-\varepsilon))$, and on L we have:

$$\text{(II.8)} \qquad |F_n(s)| \leqslant \sum(-1)^p \alpha_p^{(n)}(2p)!\,(\pi R)^{-2p}\cdot M(s) = A_n(R)M(s).$$

(It is enough to set in (II.7): $\bar{D}^{\cdot}+\varepsilon=R$.) But when we set (see I.1.2)

$$\Lambda_0(r) = \sum(-1)^p a_p r^{2p},$$

we have $(-1)^p\alpha_p^{(n)} \leqslant (-1)^p\alpha_p$, and when we write

$$C(R) = \sum(-1)^p\alpha_p(2p)!\,(\pi R)^{-2p},$$

we have $A_n(R) \leqslant C(R) < \infty$ for $R > \bar{D}^{\cdot}$. Hence

$$\text{(II.9)} \qquad |F_n(s)| \leqslant C(R)M(s).$$

Note that by setting $f_1(s)=\sum |a_n|\, e^{-\lambda_n s}$ we have another inequality similar to (II.7), where f is replaced by f_1 and $M(s)$ by $M_1(s)$, M_1 being defined like M but with f replaced by f_1, where s is taken on that part of L for which $C(s, \pi R) \subset P_{\sigma_c}$. Without loss of generality we may assume that L contains part of the real axis. When we write $F^{\cdot}_n(s)=\sum(-1)^p\alpha_p^{(n)}f_1^{(2p)}(s)$, we see that for sufficiently large $\sigma \in L$ we have:

$$\begin{aligned} F^{\cdot}_n(\sigma) &= \sum(-1)^p\alpha_p^{(n)}f_1^{(2p)}(\sigma) \\ &= \sum_p(-1)^p\alpha_p^{(n)}\sum_m \lambda_m^{2p}|a_m|\, e^{-\lambda_m\sigma}; \end{aligned}$$

and since all the terms of the double sum are positive, we can change the order of summation also in the series

$$\begin{aligned} F_n(s) &= \sum \alpha_p^{(n)}f^{(2p)}(s) = \sum \alpha_p^{(n)}\sum \lambda_m^{2p}\, a_m\, e^{-\lambda_m s} \\ &= \sum a_m\, e^{-\lambda_m s}\sum \alpha_p^{(n)}\lambda_m^{2p} = \sum a_m\Lambda_n(\lambda_m)\, e^{-\lambda_m s}, \end{aligned}$$

as long as $s \in L$, with σ sufficiently large. But since $\Lambda_n(\lambda_m)=0$ for $m \neq n$, we finally obtain

$$F_n(s) = a_n\Lambda_n(\lambda_n)\, e^{-\lambda_n s} = \Lambda_n^{-1}a_n\, e^{-\lambda_n s},$$

where $\{\Lambda_n\}$ is the sequence associated with Λ. The inequality (II.9) with $s=s_0$ gives us the conclusion of the theorem.

Here is an important immediate corollary to Theorem II.2.1.

II.2.2. *Let* $D^{\cdot}<\infty$, $\sigma_c<\infty$. *If* (A, Λ) *has an analytic continuation in a channel* $\mathfrak{C}(L;\pi R)$ *with* $R>\bar{D}^{\cdot}$ *of which* $C(s_0, \pi R)$ $(s_0=\sigma_0+it_0)$ *is a disk, then*

$$\sigma_0 \geqslant \sigma_c - \limsup \frac{\log A_n}{\lambda_n}.$$

It is sufficient to take account of I.2.5.

Taking account of I.1.1 we obtain the following statement:

II.2.3. *If the step* h *of* Λ *is positive and if the other conditions of* II.2.2 *are satisfied* ($D^{\cdot}<\infty$ holds automatically, because $h>0$), *then we have for every* $a>1$:

$$\sigma_0 \geqslant \sigma_c + [2a\log(hD^{\cdot}) - A(a)]D^{\cdot} = \sigma_c - L(a, h, D^{\cdot}),$$

where $A(a)$ *is defined by* I.1.1.

In particular, such a series (A, Λ) *cannot be continued analytically in a channel of width* $2\pi R$, $R>D^{\cdot}$, *for which the abscissa of the centre of a disk belonging to it is smaller than* $[3\log(hD^{\cdot})-8.5]D^{\cdot}+\sigma_c$.

NOTE

[1] $C(\pi R)=\pi RL(\pi R)$, where L is the Laplace transform of Λ_0 (see I.1.2).

CHAPTER III

THEOREMS OF LIOUVILLE–WEIERSTRASS–PICARD TYPE OF ARITHMETICAL CHARACTER AND OF GENERAL CHARACTER

III.1. Theorems of arithmetical type

We are going to prove the following theorem.

THEOREM III.1.1. *Suppose that* λ_k *for a given* k *has the following property: the only rational numbers* $\alpha_1, \alpha_2, \ldots, \alpha_m$ $(m>k)$, α, *with* $a_j \geqslant 0$ $(j=1, 2, \ldots, m)$ *satisfying the relation*

$$\lambda_k = \alpha_1\lambda_1 + \alpha_2\lambda_2 + \cdots + \alpha_m\lambda_m + \alpha,$$

are $\alpha_1=\cdots\alpha_{k-1}=\alpha_{k+1}\cdots=\alpha_m=0$, $\alpha_k=1$, $\alpha=0$.

Suppose that $f(s)$ *defined by* (A, Λ) *with* $\sigma_a<\infty$ *has a direct analytic continuation starting from* $P_{\sigma'}$ $(\sigma'>\sigma_a)$ *into* $\mathscr{L}D(G^*)$, *where the starred curve* G^* *shelters the curve defined by*

$$r = \cos(\theta/3) \quad (|\theta| \leqslant \pi).$$

Then the set of values taken by $f(s)$ *in* $\mathscr{L}D(G^*)$ *is everywhere dense in the disk of radius* $|a_k|$ *around the origin.*

Let a be complex and $\varepsilon>0$ such that $|f(s)-a|>\varepsilon$ in $\mathscr{L}D(G^*)$. Now $\lim f(s)=0$ as $\sigma\to\infty$, uniformly with respect to t (this is so because $\lambda_1>0$). It follows that $|a|\geqslant\varepsilon$. Next we set

$$F(s) = \frac{1}{a-f(s)} - \frac{1}{a}.$$

Then $F(s)$ is a holomorphic function in $\mathscr{L}D(G^*)$, and in this domain $|F(s)|<\varepsilon^{-1}+|a|^{-1}\leqslant 2\varepsilon^{-1}$.

If σ is such that

$$\sum |a_n|\, e^{-\lambda_n\sigma} < |a|,$$

we can write

$$F(s) = \frac{1}{a}\sum\Big(\sum a_n\, e^{-\lambda_n s}/a\Big)^m = \sum C_p\, e^{-L_p s},$$

where $\{L_n\}$ is a sequence of positive numbers increasing (strictly) to infinity.

Every L_n is of the form

$$L_n = A_1\lambda_1 + A_2\lambda_2 + \cdots + A_m\lambda_m,$$

where the A_j are positive integers.

If p is any positive integer, there exists a term L_q of $\{L_n\}$ equal to $p\lambda_k$; but according to the assumption made on λ_k, no combination

$$A_1\lambda_1 + A_2\lambda_2 + \cdots + A_m\lambda_m + A,$$

where the A_j $(j=1, \ldots, m)$ are non-negative integers and where A is an integer, can be equal to $p\lambda_k$, provided that the integers A and A_j $(j\neq k)$ are not all zero and $A_k=p$.

In other words, no difference

$$L_q - L_n \quad (n \neq q)$$

is an integer.

Since for $L_q=p\lambda_k$ we have $C_q=a_k^p/a^{p+1}$, Theorem II.1.3 shows that

$$|a_k^p| \leqslant B_1|a|^{p+1} \sup_{s\in\mathscr{L}D(G^*)} |F(s)| = 2B_1|a|^{p+1}\varepsilon^{-1}.$$

By extracting the p-th root and letting p tend to infinity we obtain

$$|a_k| \leqslant |a|,$$

which proves the theorem.

Let $M=\{\mu_p\}$ be an increasing partial sequence of Λ: $M\subset\Lambda$. We say that M is a *linearly isolated* partial sequence of Λ if the difference between any linear combination with positive integral coefficients

$$m_1\mu_1 + m_2\mu_2 + \cdots + m_p\mu_p$$

and a linear combination with positive integral coefficients

$$A_1\lambda_{n_1} + A_2\lambda_{n_2} + \cdots + A_q\lambda_{n_q}$$

is not an integer unless the combinations are identical (that is, $q=p$, $\lambda_{n_j}=\mu_j$, $A_j=m_j$ $(j=1, 2, \ldots, p)$).

The following theorem is of ‘Picard’ type, with the conditions bearing on the ‘arithmetical’ exponents.

THEOREM III.1.2. *Let* $\{\lambda_{\nu_j}\}$ *be a linearly isolated partial sequence of* Λ. *We assume that* $f(s)$ *defined by the series* (A, Λ) *with* $\sigma_a < \infty$ *has an analytic continuation starting from* $P_{\sigma'}$ $(\sigma' > \sigma_a)$ *into* $\mathscr{L}D(G^*)$, *where* G^* *is defined as in* III.1.1.

Then $f(s)$ *assumes in* $\mathscr{L}D(G^*)$ *all the values of the disk* $C(0, \sum |a_{\nu_j}|)$ $(|s| < \sum |a_{\nu_j}|)$, *except possibly the value zero and one other value.*

It is known that if $a \neq b$, there exists in the complex plane a function $\chi(w)$ with the following properties:

(α) $\chi(w)$ is regular at the origin and consequently is given in the neighbourhood of the origin by a series $\sum A_n w^n$;

(β) $\chi(w)$ has an analytic continuation on every curve that does not pass through a or b, with $w = a$ and $w = b$ as ramification points for $\chi(w)$;

(γ) $|\chi(w)| < 1$. Evidently

$$\limsup |A_n|^{1/n} = [\min(|a|, |b|)]^{-1} = c^{-1}.$$

Suppose then that, contrary to our assertion, $f(s)$ does not take the two values a and b such that $a \neq b$, $a \neq 0$, $b \neq 0$, $|a| < A = \sum |a_{\nu_j}|$, $|b| < A$, and consider the function $F(s) = \chi \circ f (= \chi(f(s))$, where χ is the function we have just defined.

If σ' is such that $\sum |a_n|\, e^{-\lambda_n \sigma'} < c$, we see that

$$\sum |A_m| \left(\sum |a_n|\, e^{-\lambda_n \sigma'}\right)^m$$

converges and consequently that $F(s)$ for sufficiently large σ is given by a Dirichlet series

$$\begin{aligned} F(s) &= \sum A_m (f(s))^m \\ &= \sum A_n \sum_{\substack{k_1 + k_2 + \cdots + k_q = n \\ 1 \leqslant l_j < \infty \\ j = 1, 2, \ldots, q}} \frac{n!}{k_1! \ldots k_q!}\, a_{l_1}^{k_1} \ldots a_{l_q}^{k_q}\, e^{-s(k_1 \lambda_{l_1} + \cdots + k_q \lambda_{l_q})} \\ &= \sum C_\nu\, e^{-L_\nu s}, \end{aligned}$$

a series having an abscissa of absolute convergence. In $\mathscr{L}D(G^*)$ we have $|F(s)| < 1$. By the condition on $\{\lambda_{\nu_j}\}$, every L_κ which is a linear combination with positive integral coefficients of the λ_{ν_j} differs from any other L_ν by a number that is not an integer. Let

$$L_\kappa = m_1 \lambda_{\nu_1} + m_2 \lambda_{\nu_2} + \cdots + m_p \lambda_{\nu_p}.$$

The corresponding coefficient c_k of $F(s)$ is given by:

$$c_k = \frac{n!A_n}{m_1!\dots m_p!}\, a_{\nu_p}^{m_1}\dots a_{\nu_p}^{m_p}, \quad (m_1 + m_2 + \cdots + m_p = n).$$

Theorem II.1.3 allows us to write

$$\text{(III.1)} \qquad \frac{n!}{m_1!\dots m_p!}\,|a_{\nu_1}|^{m_1}\dots|a_{\nu_p}|^{m_p} < \frac{B_1}{|A_n|}$$

where $p, \nu_1, \dots, \nu_p$ and n are given and the inequality is valid for any positive integers $m_1, m_2, \dots, m_p$ for which

$$m_1 + m_2 + \cdots + m_p = n.$$

We fix p and the integers $\nu_1, \nu_2, \dots, \nu_p$. We take p positive constants $\alpha_1, \dots, \alpha_p$ with $\alpha_1 + \cdots + \alpha_p = 1$, and for every positive n we set

$$\begin{aligned} m_j &= E(n\alpha_j) \quad (1 \leqslant j \leqslant p - 1),\\ m_p &= n - (m_1 + m_2 + \cdots + m_{p-1}). \end{aligned}$$

By Stirling's formula we have

$$\lim_{n} \left(\frac{n!}{m_1!\dots m_p!}\right)^{1/n} = \alpha_1^{-\alpha_1}\dots\alpha_p^{-\alpha_p}.$$

From (III.1) it follows that

$$\text{(III.2)} \qquad \begin{aligned} \alpha_1^{-\alpha_1}\dots\alpha_p^{-\alpha_p}|a_{\nu_1}|^{\alpha_1}\dots|a_{\nu_p}|^{\alpha_p} &\leqslant (\limsup |A_n|^{1/n})^{-1}\\ &= c = \min(|a|, |b|). \end{aligned}$$

We see this by setting in (III.1) $n = n_j$, where $\{n_j\}$ is a sequence for which $\lim |A_{n_j}|^{1/n_j} = \limsup |A_n|^{1/n}$, and by letting j tend to infinity.

Since we may assume that all the a_n are non-zero, we may set

$$\alpha_j = |a_{\nu_j}|/(|a_{\nu_1}| + \cdots + |a_{\nu_p}|) \quad (1 \leqslant j \leqslant p).$$

We then have $\alpha_j^{\alpha_j} = [(|a_{\nu_j}|/(|a_{\nu_1}| + \cdots + |a_{\nu_p}|)]^{\alpha_j}$, and (III.2) gives us immediately:

$$|a_{\nu_1}| + \cdots + |a_{\nu_p}| \leqslant \min(|a|, |b|).$$

Since p is arbitrary, we arrive at the contradiction

$$\sum |a_{\nu_j}| \leqslant \min(|a|, |b|) < A = \sum |a_{\nu_j}|.$$

Theorem III.1.2 is evidently more precise than Theorem III.1.1 (not only is Theorem III.1.2 of 'Picard' type, while Theorem III.1.1 is of 'Weierstrass' type, but also the radius of the 'disk of the distribution of values' is smaller in the latter proposition). But we have thought it useful to demonstrate both theorems, because although the two theorems are based essentially on Theorem II.1.3, the passage from this to Theorem III.1.1 is elementary, whereas we had to use the modular function in order to pass from Theorem II.1.3 to III.1.2.

Here is an immediate corollary to III.1.2.

III.1.3. *If the sequence* $\{\lambda_n\}$ *is such that no combination*

$$A_1\lambda_1 + A_2\lambda_2 + \cdots + A_n\lambda_n,$$

where the A_j $(j=1, \ldots, n)$ *are integers, is itself an integer, and if the conditions of* III.1.2 *on* $f(s)$ *are satisfied, then* $f(s)$ *takes in* $\mathscr{L}D(G^*)$ *all the values of the disk* $C(0, \sum |a_n|)$ *except possibly the value zero and a single other value.*

We ought to emphasize the fact that in Theorems III.1.1 and III.1.2 a single coefficient, or a subset of coefficients whose exponents are distinguished by their arithmetical character from the other exponents, can already furnish important information on the function. They give properties that cannot be annihilated by the coefficients corresponding to other exponents. In Theorem III.1.1 the object to be evaluated is the least upper bound of $|f(s)|$ in a domain whose form depends on the character of the exponent, and in Theorem III.1.2 it is the image of this domain under f. We shall see later that the distribution of singularities of f can also be determined to a large extent by the coefficients corresponding to exponents that are 'isolated' in a certain fashion.

III.2. Liouville–Picard theorems of general type

We assume that the upper density $D^{\cdot}$ on Λ is finite; we also assume now that the series (A, Λ) converges in the entire plane: $\sigma_c = -\infty$. From the fact that $D^{\cdot} < \infty$ it follows that $\sigma_c = \sigma_a = \lim \log |a_n|/\lambda_n = -\infty$.

The function $f(s)$ given by (A, Λ) is therefore entire. For every real σ we set:

$$M(\sigma) = \operatorname*{lub}_{t} |f(\sigma + it)|.$$

It follows from I.3.4 that $\lim M(\sigma)=\infty$ as σ tends to $-\infty$, provided f is not identically zero. In that case the quantity

$$\text{(III.3)}\qquad \rho = \limsup_{\sigma\to-\infty} \frac{\log\log M(\sigma)}{-\sigma}$$

is called the (R)-*order of* f (the *Ritt order of* f). One must not confuse the (R)-order of f with the order of this entire function in the classical sense, that is, the order defined by

$$\delta = \text{lin sup} \frac{\log\log \mathfrak{M}(r)}{\log r},$$

where $\mathfrak{M}(r)=\max|f(s)|(|s|=r)$. Thus, the (R)-order of $f(s)=e^{-s}$ is 0, but its order in the classical sense is 1.

The following proposition is useful:

Theorem III.2.1. *If*

$$\text{(III.4)}\qquad \liminf \frac{\lambda_n}{\log n} > 0,$$

then a necessary and sufficient condition for the (R)*-order of* f *to be equal to* ρ *is*

$$\text{(III.5)}\qquad \limsup \frac{\log|a_n|}{\lambda_n\log\lambda_n} = -\frac{1}{\rho}.$$

(The second term of this equality is $-\infty$ if the (R)-order of f is 0, and vice versa.)

Note that in the statement we do not demand that $D^{\cdot}<\infty$; but the condition (III.4) is satisfied when $D^{\cdot}<\infty$. Since in this section we assume the latter condition to be satisfied, the (R)-order of f is here always given by (III.5).

Suppose that ρ, the (R)-order of $f(s)$, is finite. For every σ we have (see (I.10)):

$$|a_n| \leqslant M(\sigma)\, e^{\lambda_n\sigma}.$$

It then follows from (III.3) that for every $\varepsilon>0$ and sufficiently large $-\sigma$ we have

$$\log|a_n| \leqslant \log M(\sigma) + \lambda_n\sigma < e^{-(\rho+\varepsilon)\sigma} + \lambda_n\sigma.$$

Now the minimum of the third term of this inequality is assumed for $\sigma = -\log[\lambda_n/(\rho+\varepsilon)]/(\rho+\varepsilon)$, a quantity that tends to $-\infty$ as n tends to $+\infty$. Hence for sufficiently large n we have:

$$\begin{aligned}\log|a_n| &\leqslant \min\,(e^{-(\rho+\varepsilon)\sigma} + \lambda_n\sigma)\\ &= \lambda_n(1 - \log(\lambda_n/(\rho+\varepsilon)))/(\rho+\varepsilon),\end{aligned}$$

which gives

$$\limsup \frac{\log|a_n|}{\lambda_n \log \lambda_n} \leqslant -\frac{1}{\rho+\varepsilon},$$

that is, since $\varepsilon > 0$ is arbitrary,

$$\limsup \frac{\log|a_n|}{\lambda_n \log \lambda_n} \leqslant -\frac{1}{\rho}.$$

Evidently this inequality is also valid for $\rho = \infty$. On the other hand, if this inequality is also valid for $\rho < \infty$, then to every $\varepsilon > 0$ we can assign a quantity $B(\varepsilon)$ such that

$$|a_n| \leqslant B(\varepsilon)\, e^{-\lambda_n \log \lambda_n/(\rho+\varepsilon)} = B(\varepsilon)\lambda_n^{-\lambda_n/(\rho+\varepsilon)},$$

which, in turn, gives

$$\begin{aligned}\text{(III.6)}\qquad M(\sigma) &\leqslant \sum |a_n|\, e^{-\lambda_n\sigma} \leqslant B(\varepsilon) \sum \lambda_n^{-\lambda_n/(\rho+\varepsilon)}\, e^{-\lambda_n\sigma}\\ &\leqslant B(\varepsilon) \max\,[e^{-\lambda_n \log \lambda_n/(\rho+2\varepsilon) - \lambda_n\sigma}] \sum e^{-a\lambda_n \log \lambda_n},\end{aligned}$$

with $a = \varepsilon/(\rho+\varepsilon)(\rho+2\varepsilon)$. But it follows from (III.4) that for a certain positive constant b:

$$\sum e^{-a\lambda_n \log \lambda_n} \leqslant \sum e^{-ab \log n \log \lambda_n} = \sum n^{-ab \log \lambda_n} < \infty.$$

Thus, (III.6) allows us to write, by setting $k = (\rho+2\varepsilon)^{-1}$

$$\begin{aligned}M(\sigma) &\leqslant C(\varepsilon) \max e^{-k\lambda_n \log \lambda_n - \lambda_n\sigma}\\ &\leqslant C(\varepsilon) \max e^{-kx \log x - x\sigma} = C(\varepsilon)\, e^{-(k+2\sigma)}\, e^{-(1+\sigma/k)}.\end{aligned}$$

So we have for sufficiently large $-\sigma$

$$\log M(\sigma) \leqslant e^{-(1/k+\varepsilon)\sigma},$$

which proves that

$$\limsup_{\sigma\to-\infty} \frac{\log\log M(\sigma)}{-\sigma} \leqslant \rho.$$

This completes the proof of III.2.1.

We are now going to define the order of a function f in a horizontal strip. Let $S=S(t_0, a)$ denote the strip $|t-t_0|<a$. Let $f(s)$ be a function holomorphic in the closed strip $|t-t_0|\leqslant a$ and let us set for every σ

$$M_S = \max_{|t-t_0|\leqslant a} |f(\sigma + it)|.$$

The quantity

$$\limsup_{\sigma\to-\infty} \frac{\log\log M_\sigma(\sigma)}{-\sigma} = \rho_S$$

is called the *order of f in the strip S.*

We can now prove the following theorem, which plays an important role.

THEOREM III.2.2. *Suppose that the step of Λ is positive and that the function f represented by (A, Λ) is entire. Then the order of f in every strip $S=S(t_0, \pi a)$ with $a>\bar{D}^{\cdot}$ is equal to the (R)-order of f (in the plane).*

Let ρ be the (R)-order of f and let ρ_S be its order in a strip $S(t_0, \pi a)$ with $\bar{D}^{\cdot}<a$. Clearly $\rho_S\leqslant\rho$. So we have to show that $\rho_S\geqslant\rho$. Also, it is sufficient to prove the theorem for $\rho>0$, because as we shall see later, for every f not identically 0 we have $\rho_S\geqslant 0$.

We set

$$M_1(\sigma') = \max_{s\in C} |f(s)|,$$

where $C=C(\sigma'+it_0, \pi a)$. It follows from Theorem II.2.1 that for every σ and every $n\geqslant 1$:

$$\text{(III.7)}\qquad \log M_1(\sigma) \geqslant \log|a_n| - \log\Lambda_n - \lambda_n\sigma - \log C(a).$$

By (III.5), to every $\varepsilon>0$ there corresponds a sequence of positive integers $\{n_j\}$ such that

$$\text{(III.8)}\qquad \log|a_{n_j}| \geqslant -\left(\frac{1}{\rho} + \varepsilon\right)\lambda_{n_j}\log\lambda_{n_j} \quad (j\geqslant 1),$$

and according to Proposition I.1.1 there exists a constant P such that

$$\text{(III.9)}\qquad \log\Lambda_n \leqslant P\lambda_n \quad (n\geqslant 1).$$

The inequalities (III.7), (III.8), and (III.9) give for $j > j_\varepsilon$ and for every σ (where j_ε is independent of σ) and a constant C independent of σ and j:

$$\text{(III.10)}\quad \log M_1(\sigma) \geqslant -\left(\frac{1}{\rho} + \varepsilon\right) \lambda_{n_j} \log \lambda_{n_j} - (P + \sigma)\lambda_{n_j} - C$$

$$\geqslant -\left(\frac{1}{\rho} + 2\varepsilon\right) \lambda_{n_j} \log \lambda_{n_j} - \sigma\lambda_{n_j}.$$

If we set in (III.10), in particular, $\sigma = -(1/\rho + 3\varepsilon) \log \lambda_{n_j}$, we obtain for $j \geqslant j_\varepsilon$:

$$\text{(III.11)}\quad \log M_1\left[-\left(\frac{1}{\rho} + 3\varepsilon\right) \log \lambda_{n_j}\right] \geqslant \varepsilon\lambda_{n_j} \log \lambda_{n_j}.$$

Now to every σ there corresponds an α with $|\alpha| \leqslant \pi a$ for which

$$\text{(III.12)}\quad M_S(\sigma + \alpha) \geqslant M_1(\sigma).$$

If we denote by α_j the value of α corresponding to

$$\sigma = -\left[\frac{1}{\rho} + 3\varepsilon\right] \log \lambda_{n_j},$$

then we have by virtue of (III.11) and (III.12):

$$\log M_S\left[-\left(\frac{1}{\rho} + 3\varepsilon\right) \log \lambda_{n_j} + \alpha_j\right] \geqslant \varepsilon\lambda_{n_j} \log \lambda_{n_j},$$

which allows us to write

$$\limsup_{\sigma \to -\infty} \frac{\log \log M_S(\sigma)}{-\sigma}$$

$$\geqslant \limsup_{j = \infty} \frac{\log \log M_S\left[-\left(\frac{1}{\rho} + 3\varepsilon\right) \log \lambda_{n_j} + \alpha_j\right]}{\left(\frac{1}{\rho} + 3\varepsilon\right) \log \lambda_{n_j} - \alpha_j}$$

$$\geqslant \limsup \log \lambda_{n_j} \Big/ \left(\frac{1}{\rho} + 3\varepsilon\right) \log \lambda_{n_j} = \left(\frac{1}{\rho} + 3\varepsilon\right)^{-1},$$

and since $\varepsilon > 0$ is arbitrary, the theorem is proved for $\rho > 0$.

By (III.7) we see that if f is not identically 0, then we have $\lim M_S(\sigma) = \infty$ $(\sigma \to -\infty)$ and necessarily $\rho_S \geqslant 0$.

The theorem is now proved in all cases.

Next we pass on to theorems generalizing the theorems of Liouville and of Picard for general Dirichlet series ($D^* < \infty$). We begin with the following definition. Let $s(u) = \sigma(u) + it(u)$ be a continuous complex function of the real variable $u \in]-\infty, +\infty[$ such that

$$\lim_{u=\infty} \sigma(u) = \infty, \quad \lim_{u=-\infty} \sigma(u) = -\infty,$$

with $t(u) = t_0$ for $\sigma \geqslant \sigma_0$, where t_0 and σ_0 are constant. If R is a positive quantity, the union of the disks $C(s(u), R)$ ($u \in]-\infty, +\infty[$) is called a *curvilinear strip of width* $2R$, *extending to* $-\infty$ (*horizontal, to the right*). We denote such a strip by $B(s(u), R)$.

The horizontal strips S which we have discussed above correspond to the case when $t(u) = t_0$ for every u.

The set of points $s(u)$ is the *central line of the strip* B. Here is a generalization of Liouville's theorem.

THEOREM III.2.3. *Suppose that* $D^* < \infty$ *and that* $\sigma_c < \infty$ *for the series* (A, Λ). *If* $f(s)$, *represented by* (A, Λ), *has an analytic continuation across a curvilinear strip* B *of width* $2\pi R$ *extending to* $-\infty$, *with* $R > \bar{D}^*$, *and if this continuation is bounded in* B, *then* f *is identically equal to zero* (*that is:* $a_n = 0$ *for* $n \geqslant 1$).

This theorem is an immediate corollary to II.2.1, where we put $s_0 = s(u)$ and then let u tend to $-\infty$. The conclusion of II.2.1 shows immediately that $a_n = 0$ for $n \geqslant 1$.

Next, here is a general theorem of 'Picard' type.

THEOREM III.2.4. *Suppose that* $D^* < \infty$ *and that* $f(s)$, *represented by* (A, Λ) *with* $\sigma_c < \infty$, *has an analytic continuation across a strip* $B(s(u), \pi R)$ *extending to* $-\infty$, *with* $R > D^*$; *suppose also that* f *is not identically zero.*

If $\{u_n\}$ *is an arbitrary sequence with* $\lim u_n = -\infty$, *then one of the statements* (1), (2) *holds:*

(1) $f(s)$ *assumes in the set of disks* $C(s(u_n), \pi R)$ *all values except possibly one.*

(2) *For every* ε *with* $0 < \varepsilon < R$ *we have*

$$\lim f(s) = \infty$$

uniformly (*with respect to* s) *as long as* s *remains in the disks*

$$C(s(u_n), \pi(R - \varepsilon)),$$

and n *tends to infinity.*

If $s(u)$ *is real for* $u<u'$ $(s(u)=\sigma(u))$, *we can replace in this statement the disks* $C(s(u_n), \pi R)$, $C(s(u_n), \pi(R)-\varepsilon))$ *by the squares*

$$|\sigma - \sigma(u_n)| < \pi R, \qquad |t| < \pi R;$$
$$|\sigma - \sigma(u_n)| < \pi(R - \varepsilon), \qquad |t| < \pi(R - \varepsilon).$$

We consider the family $\mathfrak{F}$ of holomorphic functions

$$f_n(s) = f(s + s(u_n))$$

defined in the disk $C \equiv C(0, \pi R)$. If (1) does not hold, the family is normal in C. But no sequence extracted from $\mathfrak{F}$ can be bounded in C, because then, by Theorem II.2.1, we would have $f(s) \equiv 0$; hence, if (1) does not hold, then we can extract from every sequence $f_n(s)$ a sequence tending to infinity uniformly for every closed disk in the interior of C. It then follows that (2) holds. The last part of the theorem is proved similarly.

The following lemmas will be used later.

III.2.5. *Let* $\mathfrak{F}$ *be a family of functions* $\varphi(s)$ *holomorphic in a domain* $\mathfrak{D}$ *and such that:* $|\varphi(s)|>1$ *in* $\mathfrak{D}$. *To every bounded domain* $\mathfrak{D}_1$ *for which* $\bar{\mathfrak{D}}_1 \subset \mathfrak{D}$ *there corresponds a constant* α *such that*

$$\frac{1}{\alpha} < \frac{\log|\varphi(s_1)|}{\log|\varphi(s_2)|} < \alpha$$

where $\varphi \in \mathfrak{F}$; $s_1, s_2 \in \bar{\mathfrak{D}}_1$.

For, the family of harmonic functions

$$\Psi_{s_2}(s) = \frac{\log|\varphi(s)|}{\log|\varphi(s_2)|},$$

where φ is an arbitrary function of $\mathfrak{F}$ and where s_2 is an arbitrary point of $\bar{\mathfrak{D}}_1$, is normal in $\mathfrak{D}$, because all the functions $\Psi_{s_2}(s)$ are positive in $\mathfrak{D}$. This proves that this family is bounded on $\bar{\mathfrak{D}}_1$, because $\Psi_{s_2}(s_2)=1$ for every $s_2 \in \bar{\mathfrak{D}}_1$. To obtain the lemma it suffices to make the same remark with s_2 replaced by s_1.

III.2.6. *Let* a *and* b *be two constants with* $0<a<b$, *and let* $F(s)$ *be a function holomorphic in the horizontal strip* $S(t_0, b)$. *If the order of the strip* $S(t_0, a)$ *is infinite, the function cannot tend to infinity uniformly in* $S(t_0, b)$ *as* σ *tends to* $-\infty$.

Suppose, contrary to our statement, that $F(s)$ tends to infinity uniformly in $S(t_0, b)$ as $\sigma \to -\infty$. Consider the family $\mathfrak{F}$ of functions $F_d(s) = F(s+d)$, where d is an arbitrary negative quantity and s varies in the square Δ_1: $|\sigma - \sigma_0| < b$, $|t - t_0| < b$. We have

$$\lim_{d=-\infty} F_d(s) = \lim_{d=-\infty} F(s+d) = \infty$$

uniformly in s for $s \in \Delta_1$. And if σ_0 is chosen negative and of sufficiently large absolute value, then for every $d \leqslant 0$, $s \in \Delta_1$, we have: $|F_d(s)| > 1$. Let $n(l)$ be the integral part of $-l/2a$. Applying Lemma III.2.5 to the family $F_d(s)$ $(d < 0)$, we see that there exists a constant $\alpha > 1$ such that for any s and s' with $s, s' \in \bar{\Delta}_2$, where $\bar{\Delta}_2$ is the closed square $|\sigma - \sigma_0| \leqslant a$, $|t - t_0| \leqslant a$, we have

$$\text{(III.13)} \quad \frac{\log |F_d(s)|}{\log |F_d(s')|} < \alpha.$$

Now let σ_1 be a negative quantity of sufficiently large absolute value, and let us write down the inequality (III.13) for d, s, and s' chosen in the following manner:

(1) $d = \sigma_1$, s is arbitrary in $\bar{\Delta}_2$, $s' = \sigma_0 + a$;

(2) $d = \sigma_1 + 2ka$ $(1 \leqslant k \leqslant n(\sigma_1))$, $s = \sigma_0 - a$, $s' = \sigma_0 + a$;

(3) $d = 0$, $s = \sigma_0 + a + 2an(\sigma_1) + \sigma_1$, $s' = \sigma_0 - a$.

Since

$$F_{\sigma_1 + 2ka}(\sigma_0 + a) = F_{\sigma_1 + 2(k+1)a}(\sigma_0 - a),$$

we obtain for $s \in \bar{\Delta}_2$ by multiplying these inequalities:

$$\begin{aligned} \log |F(\sigma_1 + s)| &= \log |F_{\sigma_1}(s)| \\ &\leqslant a^{n(\sigma_1)+1} \log |F(\sigma_1 + \sigma_0 + a + 2n(\sigma_1)a)| \\ &\leqslant \alpha^{n(\sigma_1)+2} \log |F(\sigma_0 - a)| \\ &\leqslant \alpha^{2-\sigma_1/2a} \log |F(\sigma_0 - a)|. \end{aligned}$$

It now suffices to recall the definition of the order in a horizontal strip to see that for $S = S(t_0, a)$

$$\rho_S \leqslant \log \alpha/2a < \infty,$$

which contradicts our assumption. Thus, our lemma is proved.

By combining III.2.2, III.2.4, and III.2.6 we obtain the following theorem:

THEOREM III.2.7. *Suppose that the step of Λ is positive and that $f(s)$, represented by (A, Λ), is an entire function of infinite (R)-order. Then f assumes all values, except possibly one, infinitely often in every horizontal strip $S(t_0, \pi a)$ with $a > \bar{D}^{\cdot}$.*

Note that Lemma III.2.6 is a particular case of the theorem of Bieberbach-Valiron, which can be stated in the following form:

III.2.8. *Let a and b be such that $0 < a < b$. If the function $F(s)$ is holomorphic in the strip $S(t_0, \pi a)$ and if the order of F in $S(t_0, \pi a)$ exceeds $1/2a$, then the function assumes in every half-strip*

$$|t - t_0| < \pi b, \qquad \sigma < \sigma_1,$$

all values, except possibly one.

For a proof of this theorem we refer the reader to Valiron [59].

By combining Theorem III.2.2 with III.2.8 we obtain the following theorem:

THEOREM III.2.9. *If Λ has positive step, and if ρ is the order of the entire function $f(s)$ given by (A, Λ) $(\sigma_c = -\infty)$, then f assumes in every horizontal strip $S(t_0, \pi a)$ with $a > \max(\bar{D}^{\cdot}, 1/2\,\rho)$ all the values, except possibly one, infinitely often.*

CHAPTER IV

SINGULARITIES OF FUNCTIONS REPRESENTED BY A TAYLOR SERIES

IV.1. Singularities of Taylor Series

This section is concerned with the case when $\lambda_n = n$, and instead of writing the series in the form of a Taylor-D series: $\sum a_n e^{-ns}$, we write it in the usual form: $\sum a_n z^n$ $(z = x + iy)$. There exists an enormous literature on the analytic continuation of a Taylor series, the distribution of its singularities, their nature, etc. Nevertheless we come back to this subject, because we wish, first of all, to give an account of a general theorem which plays for the argument of the singularities (on the circle of convergence) a similar role to that of Hadamard's theorem for their modulus (the formula of Cauchy–Hadamard providing the radius of convergence). Our theorem is very general and has a considerable number of applications. Since the Taylor-D series is a very important case of the general Dirichlet series, we have thought it useful to devote the present chapter to this subject.

Here is the theorem on the arguments of the singularities.

THEOREM IV.1.1. *Let* R $(0 < R < \infty)$ *be the radius of convergence of the series*

$$\text{(IV.1)} \qquad F(z) = \sum a_n z^n,$$

and let us set for $h \geqslant 0$:

$$D(h) = \limsup |a_0 h^n + \binom{n}{1} a_1 h^{n-1} + \cdots + a_n|^{1/n}.$$

Then $D(h)$ *is a continuous function at* $h = 0$ *and has at* $h = 0$ *a right-hand derivative*

$$D'_+(0) = \lim_{h \downarrow 0} \frac{D(h) - D(0)}{h},$$

for which

$$|D'_+(0)| \leqslant 1;$$

and if we write

$$D'_+(0) = \cos\varphi,$$

then one of the points $Re^{i\varphi}$ *or* $Re^{-i\varphi}$ *is a singularity of* (IV.1), *namely the one that is nearest to* $z=R$ *on the circle of convergence.*[1]

Consider the series

$$\text{(IV.2)}\qquad g_0(z) = \sum \frac{a_n}{z^n}.$$

If z_0 is a singularity of $F(z)$, then z_0^{-1} is a singularity of $g_0(z)$, and vice versa.

The radius of convergence of (IV.2) is $R^{-1}=D(0)$.

Since the function $g_0(z+h)/z$ is regular at infinity, we can write for sufficiently large $|z|$

$$\text{(IV.3)}\qquad g_0(z)\frac{z+h}{z} = \sum_{n\geqslant 0} \frac{d_n(h)}{(z+h)^n} = g_h(z),$$

with

$$\begin{aligned} d_n(h) &= \frac{-1}{2\pi i}\oint \frac{g_0(z)}{z}(z+h)^n\,dz \\ &= \frac{-1}{2\pi i}\oint \frac{g_0(z)}{z}\left(z^n + \binom{n}{1} z^{n-1}h + \cdots + h^n\right) dz \\ &= a_n + \binom{n}{1} a_{n-1}h + \cdots + a_0h^n. \end{aligned}$$

The radius of convergence of the series (IV.3) is equal to $D(h)$. Now $g_0(z)$ and $g_h(z)$ have the same singularities, except possibly the point $z_0=-h$, which may be a singularity for $g_0(z)$ without being one for $g_h(z)$.

We denote by $S^+(h)$ and $S^-(h)$ the domains $|z+h|>D(h)$ and $|z+h|<D(h)$, respectively, and by $C(h)$ their common boundary. Since $g_h(z)$ has all its singularities in $\bar{S}^-(h)=S^-(h)\cup C(h)$, and since there is at least one singularity of $g_h(z)$ on $C(h)$, we see that $D(h)$ is positive for sufficiently small h, that the intersection $I(h)=S^-(h)\cap S^-(0)$ is not empty, and that all the singularities of $g_0(z)$ are on $\bar{I}(h)$. If $z_0=R$ is

a singular point of $F(z)$, then $z=D(0)$ is a singularity of $g_0(z)$, and since $h \geqslant 0$, we have:

$$D(h) = D(0) + h.$$

In this case the theorem is therefore trivial. Suppose then that $z=R$ is a regular point of $F(z)$, so that $z=D(0)$ is a regular point for $g_0(z)$. Since $I(z)$ is not empty, we have

$$D(0) - h \leqslant D(h) \leqslant D(0) + h.$$

In other words, $D(h)$ is a continuous function at $h=0$. We denote by $Re^{i\varphi}$ the singular point of $F(z)$ on $|z|=R$ whose argument has the smallest absolute value (or one of these two points, if both $Re^{i\varphi}$ and $Re^{-i\varphi}$ are singular points for $F(z)$). The point $q=D(0)e^{-i\varphi}$ is the singular point of $g_0(z)$ on $|z|=D(0)$ with the smallest absolute value of the argument. We denote by $A(h)$ the arc of $C(h)$ in the interior of $C(0)$. Since $g_h(z)$ has no singularities on $C(h)-\bar{A}(h)$, there is at least one singularity of $g_h(z)$ on $\bar{A}(h)$. But $g_0(z)$ has no singularity on the part of $C(0)$ exterior to $C(h)$. Since the points $-h+D(h)e^{i\theta}$ of $C(h)$ tend to the points of $C(0)$, uniformly in θ, as the points of $A(h)$ tend to the part of the arc $D(0)e^{i\psi}$ where $|\psi| \leqslant \varphi$, and since the set of singularities is closed, we see immediately that the points of intersection of $C(0)$ with $C(h)$ tend, respectively, to $D(0)e^{i\varphi}$ and $D(0)e^{-i\varphi}$.

If $x_h \pm iy_h$ are the points of intersection of the circles $C(0)$ and $C(h)$, we have:

$$x_h = \frac{D^2(h) - D^2(0)}{2h} - \frac{h}{2},$$

and by what we have just seen,

$$\begin{aligned}\lim_{h\downarrow 0} x_h &= \lim_{h\downarrow 0} \frac{D^2(h) - D^2(0)}{2h} \\ &= D(0) \lim_{h\downarrow 0} \frac{D(h) - D(0)}{h} = D(0) \cos \varphi,\end{aligned}$$

which proves the theorem.

We are now going to show that we can preserve the form of the terms participating in the formation of $d_n(h)$, but neglect a certain number of them, depending on n and h; this makes the formula more manageable and more susceptible to yield interesting results. So we prove here a theorem on the distribution of the singularities of $F(z)$ on its circle of convergence, which is of a rather curious character. Later we shall try to clarify the difference between this proposition and the classical 'lacunary' propositions.

We begin with some lemmas.

IV.1.2. *Let* $\limsup |a_n|^{1/n}=1$ *and let* $\alpha\leqslant 0$, $\lambda>0$ *be such that*

$$\text{(IV.5)} \qquad \alpha > \frac{\lambda}{1+\lambda} - \log(1+\lambda).$$

Let $\{u_j\}$ *be a sequence of positive numbers tending to zero and suppose that to every* j *there corresponds a sequence* $\{m_n^j\}$ *of positive integers such that*

$$\text{(IV.6)} \qquad \limsup_{n\to\infty} |a_{m_n^j}|^{1/m_n^j} \geqslant 1 + \alpha u_j + o(u_j) \quad (j\to\infty)$$

and that

$$a_m = 0, \quad \text{for} \quad m \in\,]m_n^j(1-\lambda u_j),\, m_n^j(1+u_j)[, \quad m \neq m_n^j.$$

We set $h_j=u_je^{-\alpha}$, $v_n^j=E[m_n^j(1+u_j)]$ ($E[a]$ *denoting the integral part of* a); *if* $d_n(h)$ *is defined by*

$$\text{(IV.7)} \qquad d_n(h) = a_0h^n + \binom{n}{1} a_1h^{n-1} + \cdots + a_n,$$

then

$$\limsup_{n} |d_{v_n^j}(h_j)|^{1/v_n^j} \geqslant 1 + e^{\alpha}h_j + o(h_j) \quad (j\to\infty).$$

Remark. The condition (IV.5) holds, in particular, if α and λ satisfy one of the following conditions:

(1) $\lambda = 1, \alpha > \frac{1}{2} - \log 2$

(2) $\alpha = 0, \lambda > 0$ arbitrary.

To prove IV.1.2 we first have to demonstrate the following two lemmas.

IV.1.3. *Let* $\limsup |a_n|^{1/n}=1$, $\delta>1$, *and let* $\{p_n\}$ *and* $\{q_n\}$ *be two sequences of positive integers tending to infinity. If for* $h>0$:

$$\text{(IV.9)} \qquad \limsup\,(q_n/p_n) \leqslant 1 - \delta h,$$

then

$$\text{(IV.10)}\quad \limsup \left|a_0h^{p_n} + \binom{p_n}{1} a_1h^{p_n-1} + \cdots + \binom{p_n}{q_n} a_{q_n}h^{p_n-q_n}\right|^{1/p_n} \leqslant 1 + \delta(1-\log\delta)h + o(h) \quad (h \to 0).$$

We set

$$\delta_n(h) = a_0h^{p_n} + \binom{p_n}{1} a_1h^{p_n-1} + \cdots + \binom{p_n}{q_n} a_{q_n}h^{p_n-q_n}.$$

We see, first of all, that for every $a > 1$

$$\limsup |\delta_n(h)|^{1/p_n} \leqslant \limsup \left\{1 + \binom{p_n}{1}\left(\frac{a}{h}\right) + \cdots + \binom{p_n}{q_n}\left(\frac{a}{h}\right)^{q_n}\right\}^{1/p_n} h.$$

Also for every $b > 0$:

$$P_n(w) = 1 + \binom{p_n}{1} w + \cdots + \binom{p_n}{q_n} w^{q_n} = \frac{1}{2\pi i}\oint \left(1 + \frac{w}{z}\right)^{p_n}(1 + z + \cdots + z^{q_n})\,\frac{dz}{z}$$

where the integral is taken over $|z| = b$; consequently for $b > 1$:

$$|P_n(w)| \leqslant \left(1 + \frac{|w|}{b}\right)^{p_n} q_nb^{q_n},$$

which allows us to write, when (IV.9) holds:

$$\limsup |P_n(w)|^{1/p_n} \leqslant (b + |w|)b^{-\delta h}.$$

It is sufficient to set $w = h/a$, $a > 1$, $b = \delta$ with $a \downarrow 1$ to obtain (IV.10).

IV.1.4. *With the notation and conditions of Lemma* IV.1.3, *let* $c > \delta(1 - \log \delta)$. *If*

$$\limsup \left|\binom{p_n}{q_n} h^{p_n-q_n} + \cdots + a_{p_n}\right|^{1/p_n} \geqslant 1 + ch + o(h).$$

and if $d_n(h)$ *is defined by* (IV.7), *then:*

$$\limsup |d_{p_n}(h)|^{1/p_n} \geqslant 1 + ch + o(h) \quad (h \to 0).$$

This lemma is an immediate consequence of Lemma IV.1.3 (where q_n is replaced by $q_n - 1$).

We now prove Lemma IV.1.2. Let $\varepsilon > 0$ and for fixed j let:

$$p_n = E[m_n^j(1 + u_j)],\ q_n = E[m_n^j(1 - \lambda u_j)] \quad (n \geqslant 1).$$

When j is sufficiently large (u_j sufficiently small), we have on the one hand:

$$\text{(IV.11)} \quad \limsup \frac{q_n}{p_n} = \frac{1 - \lambda u_j}{1 + u_j} = 1 + (1 + \lambda)u_j + o(u_j)$$
$$= 1 - (1 + \lambda)e^\alpha h_j + o(h_j)$$
$$\leqslant 1 - \{(1 + \lambda)e^\alpha - \varepsilon\}h_j,$$

and on the other hand:

$$\limsup \left| \binom{p_n}{q_n} a_{q_n} h^{p_n - q_n} + \cdots + a_{p_n} \right|^{1/p_n}$$
$$= \limsup \left| \binom{p_n}{m_n^j} a_{m_n^j} h^{p_n - m_n^j} \right|^{1/p_n}$$
$$\geqslant (1 + u_j)\left(\frac{h_j}{u_j}\right)^{u_j/(1 - u_j)} (1 + \alpha u_j + o(u_j))^{1/(1 + u_j)}$$
$$= 1 + e^\alpha h_j + o(h_j) \quad (j \to \infty).$$

Since it follows from (IV.11) that for sufficiently large j:

$$\limsup \frac{q_n}{p_n} \leqslant 1 - \delta h_j \quad (\delta = (1 + \lambda)e^\alpha - \varepsilon),$$
$$c = e^\alpha > \delta(1 - \log \delta),$$

the conclusion of Lemma IV.1.2 follows from Lemma IV.1.4.

We now prove the following theorem.

THEOREM IV.1.5. *Let*

$$F(z) = \sum a_n z^n$$

be a Taylor series with radius of convergence 1 and $\lambda > 0$, $0 \geqslant \alpha > \lambda/(1 + \lambda) - \log(1 + \lambda)$. *Let* $\{h_j\}$ *be a positive sequence tending to zero and suppose that to every* j *there corresponds a sequence of positive integers* $\{m_n^j\}$ *and a sequence of positive numbers* $\{\beta_n^j\}$ *such that*[2]

(1) $\Re(a_m\, e^{i\beta_n^j}) \geqslant 0 \quad \text{for} \quad m \in](1 - \lambda h_j)m_n^j, (1 + h_j)m_n^j[$

(2) $\limsup |\Re(a_{m_n^j}\, e^{i\beta_n^j})|^{1/m_n^j} \geqslant 1 + \alpha h_j + o(h_j) \quad (j \to 0).$

Then the series $F(z)$ has at least one singularity at $e^{i\theta}$ with

$$|\theta| \leqslant |\arc\cos e^{\alpha}| \quad \left(|\arc\cos e^{\alpha}| \leqslant \frac{\pi}{2}\right).$$

COROLLARY to IV.1.5. *Let F, α, λ be defined as in* IV.1.5, *let $\{u_j\}$ be a sequence tending to zero, and suppose that to every j there corresponds a sequence of positive integers $\{m_n^j\}$ such that*

$$\text{(IV.12)} \quad \limsup |a_{m_n^j}|^{1/m_n^j} \geqslant 1 + \alpha u_j + o(u_j) \quad (j \to \infty)$$

$$a_m = 0 \quad \text{for} \quad m \in](1 - \lambda h_j)m_n^j, (1 + h_j)m_n^j[, \quad m \neq m_n^j.$$

Under these conditions there exists a singularity of F on every arc of the circle $|z| = 1$ with an angle $2|\arc\cos e^{\alpha}|$, $(|\arc\cos e^{\alpha}| \leqslant \pi/2)$.

To prove Theorem IV.1.5 it suffices to set

$$v_n^j = E[m_n^j(1 + u_j)], \qquad h_j = u_j\, e^{-\alpha}$$

and to note that by Lemma IV.1.2:

$$\begin{aligned} &\limsup |d_{v_n^j}(h_j)|^{1/v_n^j} \\ &\quad \geqslant \limsup \left| \Re\left((a_0 h_j^{v_n^j} + \binom{v_n^j}{1} a_1 h^{v_n^j - 1} + \cdots + a_{v_n^j})\, e^{i\beta_n^j} \right) \right|^{1/v_n^j} \\ &\quad \geqslant 1 + e^{\alpha} h_j + o(h_j); \end{aligned}$$

this proves that when we denote by $e^{i\varphi}$ the singularity of $F(z)$ on $|z| = 1$ nearest to $z = 1$ we have

$$\cos\varphi = D'_{+}(0) = \lim_{h \downarrow 0} \frac{D(h) - 1}{h} \geqslant e^{\alpha}.$$

This proves the theorem, at least for the singularities situated on an arc of angle $2|\arc\cos e^{\alpha}|$ centered at $z = 1$. An arbitrary rotation proves the rest.

Now we indicate the essential difference between Theorem IV.1.5 (and its corollary) on the one hand, and the older theorems in the same range of ideas on the other.

In the classical theorems one studies the behaviour of the coefficients a_m when the index m varies in intervals around indices m_n corresponding to 'major' coefficients, that is, those for which $\lim |a_{m_n}|^{1/m_n} = 1$ (where, for example, all the other a_m are zero, as this is the case in the Corollary to Theorem IV.1.5), the length of these intervals being proportional to m_n with a fixed proportionality factor. In contrast, here we have just introduced intervals around the indices m_n^j without assuming any sequence $\{a_{m_n^j}\}$ (for fixed j) to be 'major' (see (IV.12), where the proportionality factors of the length of these intervals decreases, and where the 'importance' of the sequence $\{a_{m_n^j}\}$ 'increases', with decreasing u_j in (IV.12), α being assumed to be non-negative). This proportionality factor $(\lambda+1)h_j$ tends to zero as j tends to infinity, whereas the expression

$$\limsup_{n} |a_{m_n^j}|^{1/m_n^j}$$

(in (IV.12)) can never be equal to 1 (if $\alpha<0$), but tends to 1 as j tends to infinity.

We end this chapter with the following theorem, which is also essentially based on Theorem IV.1.1.

THEOREM IV.1.6. *If the radius of convergence of* (IV.1) *is* 1, *if*

$$\lim |a_{n_j}|^{1/n_j} = 1,$$

and if there exists a constant $\delta>0$ *such that no polynomial of the sequence*

$$d_{n_j}(z) = a_0 z^{n_j} + \binom{n_j}{1} a_1 z^{n_j-1} + \cdots + a_{n_j} \quad (j \geqslant 1)$$

vanishes in the disk $|z|<\delta$, *then* $F(z)$ *has at least one singular point at* $e^{i\varphi}$ *with*

$$\cos\varphi \geqslant \limsup \Re \frac{a_{n_j-1}}{a_{n_j}}.$$

We see from this theorem that if $\limsup \Re(a_{n_j-1}/a_{n_j})=1$, when the other conditions are satisfied, then $z=1$ is a singular point of (IV.1).

In the notation of Theorem IV.1 we have for $h\geqslant 0$,

$$\text{(IV.13)}\quad \log D(h) = \limsup \frac{\log |d_n(h)|}{n} \geqslant \limsup \frac{\log |d_{n_j}(h)|}{n_j}.$$

There exist two constants $k>0$ and $L>0$ such that $|a_n|<Lk^n$ $(n\geqslant 0)$; consequently

$$\begin{aligned} |d_n(z)| &= \left|a_0 z^n + \binom{n}{1} a_1 z^{n-1} + \cdots + a_n\right| \\ &\leqslant L\left(|z|^n + \binom{n}{1} k|z|^{n-1} + \cdots + k^n\right) = L(|z|+k)^n. \end{aligned}$$

Since $d_{n_j}(z)$ does not vanish in $|z| < \delta$, we can define $\arg d_{n_j}(z)$ in this disk in a continuous manner, by defining $\arg d_{n_j}(0)$ equal to $\arg a_{n_j}$, the latter lying in $[0, 2\pi[$. The functions $\log d_{n_j}(z) = \log |d_{n_j}(z)| + i \arg d_{n_j}(z)$ are then holomorphic in $|z| < \delta$, and their real parts satisfy the inequality

$$\Re \log d_{n_j}(z) = \log |d_{n_j}(z)| \leqslant \log L + n_j \log (k + \delta).$$

The real parts of the functions

$$\varphi_{n_j}(z) = \frac{\log d_{n_j}(z)}{n_j}$$

are bounded. These functions therefore form a normal family in $|z| < \delta$. Consequently, we can extract from $\{n_j\}$ a sequence $\{m_p\}$ having the following two properties:

(1) $\lim \Re \dfrac{a_{m_p-1}}{a_{m_p}} = \limsup \Re \dfrac{a_{n_j-1}}{a_{n_j}} = a;$

(2) the sequence $\varphi_{m_p}(z) = \log d_{m_p}(z)/m_p$ tends in $|z| \leqslant \delta/2$ uniformly to a function holomorphic in this disk, because we have

$$\lim \varphi_{m_p}(0) = \lim \frac{\log a_{m_p}}{m_p} = \lim \frac{\log |a_{m_p}| + i \arg a_{m_p}}{m_p} = 0.$$

On the other hand, it is easy to show that

$$d_n'(z) = n d_{n-1}(z),$$

and that in $|z| < \delta$

$$\log d_n(z) = \log a_n + \int_0^z \frac{d_n'(t)}{d_n(t)}\, dt = \log a_n + n \int_0^z \frac{d_{n-1}(t)}{d_n(t)}\, dt.$$

Hence, by (IV.13) we have for $0 \leqslant h \leqslant \delta/2$:

$$\begin{aligned} \log D(h) &\geqslant \lim \frac{\log |d_{m_p}(h)|}{m_p} \\ &= \lim \left(\frac{\Re \log a_{m_p}}{m_p} + \Re \int_0^h \frac{d_{m_p-1}(t)}{d_{m_p}(t)}\, dt \right). \end{aligned}$$

But since

$$\lim \frac{\Re \log a_{m_p}}{m_p} = \lim \frac{\log |a_{m_p}|}{m_p} = 0,$$

we have

$$\log D(h) \geqslant \lim \mathfrak{R} \int_0^h \frac{d_{m_p-1}(t)}{d_{m_p}(t)}\, dt.$$

Since $D'_+(0)$ exists (Theorem IV.1.1) and from the fact that $D(0) = \limsup |a_n|^{1/n} = 1$, we also have

$$\begin{aligned} D'_+(0) &= (\log D(h))'_{+(h=0)} = \lim_{h\downarrow 0} \frac{\log D(h)}{h} \\ &\geqslant \lim_{h\downarrow 0} \frac{1}{h}\left(\lim_{p\to\infty} \mathfrak{R} \int_0^h \frac{d_{m_p-1}(t)}{d_{m_p}(t)}\, dt\right) \\ &= \lim_{p\to\infty}\left(\lim_{h\downarrow 0} \frac{1}{h} \mathfrak{R} \int_0^h \frac{d_{m_p-1}(t)}{d_{m_p}(t)}\, dt\right) \\ &= \lim \mathfrak{R} \frac{d_{m_p-1}(0)}{d_{m_p}(0)} = \lim \mathfrak{R} \frac{a_{m_p-1}}{a_{m_p}} = a \\ &= \limsup \mathfrak{R} \frac{a_{n_j-1}}{a_{n_j}}. \end{aligned}$$

But by Theorem IV.1.1 we know that (IV.1) has a singularity $e^{i\varphi}$ with $\cos \varphi = D'_+(0)$. This completes the proof of our theorem.

NOTES

[1] We recall that a 'singularity of a Taylor series on its circle of convergence' is defined as a point z_0 with $|z_0| = R$ such that for no $\varepsilon > 0$ there exists a function holomorphic in the disk $C(z_0, \varepsilon)$ that is equal to the series in $C(0, R)$.

[2] $\mathfrak{R}(a)$ denotes the real part of a.

CHAPTER V

COMPOSITION OF SINGULARITIES

V.1. Composition of Hadamard type and generalizations

Without doubt, one of the most important theorems in the theory of the distribution of singularities of a Taylor series is that of Hadamard concerning the multiplication of singularities, which he discovered in 1898.

Since later in this chapter we shall have occasion to state the theorem in its general form, we give it here under a form (due to Hadamard) which is, in fact, only valid when the singularities are isolated and the functions themselves are uniform.[1]

If E_i, $i=1, 2, 3$, *is the set of singularities of the functions represented by* $\sum a_n z^n$, $\sum b_n z^n$, *and* $\sum a_n b_n z^n$, *respectively, then to every point* e_3 *of* E_3 *there corresponds a point* e_1 *of* E_1 *and a point* e_2 *of* E_2 *such that* $e_3=e_1 e_2$.

Translated into Taylor-D series this statement becomes the following.

If γ is a singularity of the function represented by $\sum a_n b_n\, e^{-ns}$, then there is a singularity α of the function represented by $\sum a_n\, e^{-ns}$ and a singularity β of the function represented by $\sum b_n\, e^{-ns}$ such that $\gamma=\alpha+\beta$.

We assume in these statements that the radii of convergence of the series $\sum a_n z^n$ and $\sum b_n z^n$ are finite and non-zero, or what comes to the same, that the abscissae of convergence of the Taylor-D series are finite.

The statement on the Taylor-D series ceases to be true for general Dirichlet series. In other words, the function represented by $\sum a_n b_n\, e^{-\lambda_n s}$ can have singularities other than those of the form $\alpha+\beta$, where α is a singularity of the function f represented by $\sum a_n\, e^{-\lambda_n s}$, and β is a singularity of the function φ represented by $\sum b_n\, e^{-\lambda_n s}$. This remark is valid even if each of the functions f and φ has only one singularity and the functions themselves are uniform. For example, if $a_n=b_n=1$ $(n\geqslant 1)$, and $\lambda_n=\log n$, then both f and φ are the Riemann zeta-function $\zeta(s)$, which has no singularity other than a simple pole at $s=1$. Hence, if the theorem were true, the function represented by the 'composite series' $\sum a_n b_n\, e^{-\lambda_n s}$ could not have singularities other than the point $s=2$. But of course, the composite function is still $\zeta(s)$.

The fact that the statement given for Taylor-D series cannot be 'transferred' directly to general Dirichlet series is a challenge to search for a general theorem of a type *depending on properties of* Λ, which for $\Lambda = N$ $(\lambda_n = n)$ turns into that on the Taylor-D series.

We give here several theorems on the composition of singularities of Dirichlet series, which all contain as a particular case Hadamard's theorem corresponding to $\lambda_n = n$, each of the theorems having its specific character of generality.

Although the theorems we state are valid for multiform functions (after the terminology has been made somewhat more precise), to simplify the language we assume that they are uniform – at least 'uniform in a half-plane' – an expression to be made explicit later. However, we shall see that these restrictions can be removed.

So let us consider the series $(A, \Lambda) = \sum a_n e^{-\lambda_n s}$ with $\sigma_a < \infty$, and let $\sigma_1 > -\infty$. Let Δ be a domain contained in P_{σ_1} and containing $P_{\sigma'} (P_{\sigma'} \subset \Delta \subset P_{\sigma_1})$ for a certain $\sigma' > \max(\sigma_1, \sigma_a)$, and such that the sum $f(s)$ of (A, Λ) has a uniform analytic continuation from $P_{\sigma'}$ to Δ. Given σ_1, if there exists a domain, which we denote by $\Delta(\sigma_1)$, containing all the other Δ having the properties listed, we say that the function f, which is holomorphic in $\Delta(\sigma_1)$ and is there the continuation of (A, Λ) starting from $P_{\sigma'}$, *is uniform in* P_{σ_1}. The set consisting of all the points of $\bar{P}_{\sigma_1}$ that do not belong to $\Delta(\sigma_1)$ is denoted by S_{σ_1}. We call this set the *singular set of f relative to the half-plane* P_{σ_1}. The points on the line $\sigma = \sigma_1$ evidently belong to S_{σ_1}. The theorems will not be interesting unless S_{σ_1} contains points not on the line $\sigma = \sigma_1$, but for the results we shall give later to be always valid we have to define the singular set in the manner indicated.

If $\sigma_1' < \sigma_1$, the analytic continuations of (A, Λ) into $\Delta(\sigma_1)$ on the one hand, and into $\Delta(\sigma_1')$ on the other (if the two continuations are uniform) take in $\Delta(\sigma_1') \cap \Delta(\sigma_1)$ the same value; it is therefore not inconvenient to denote such a continuation (into a $\Delta(\sigma_1)$) by the same letter $f(s)$ as the sum of (A, Λ) in P_{σ_a}. Note that $\Delta(\sigma_1) \subset \Delta(\sigma_1')$. To say then that '$f(s) = (A, \Lambda) = \sum a_n e^{-\lambda_n s}$ is uniform in the half-plane P_{σ_1}' means that $\Delta(\sigma_1)$ exists, that f is holomorphic in $\Delta(\sigma_1)$, that f is given by (A, Λ) in the part of $\Delta(\sigma_1)$ belonging to P_{σ_c}, and that f has no analytic continuation in a neighbourhood of a point S_{σ_1} except along a curve intersecting the line $\sigma = \sigma_1$.

A general Dirichlet series does not necessarily have singularities on the axis of convergence, in contrast to a standard fact for the Taylor-D series (a Taylor series has at least one singular point on its circle of convergence; hence a Taylor-D series has at least one singular point on every segment of length 2π on its axis of convergence). For example, the function represented by the series $\sum(-1)^n n^{-s}$ is entire, although $\sigma_c=0$ for this series.

Now let H be the greatest lower bound of the quantities σ' such that $f(s)=(A, \Lambda)$ is holomorphic in $P_{\sigma'}$. This H is called the *abscissa of holomorphy* of (A, Λ). Evidently $H \leqslant \sigma_c \leqslant \sigma_a$. The line $\sigma=H$ is the *axis of holomorphy* of (A, Λ).

The quantity

$$H(\sigma_1) = \sup_{s\in S_{\sigma_1}} \sigma, \quad (s = \sigma + it)$$

is called the *abscissa of holomorphy* of (A, Λ) *in the half-plane* P_{σ_1}. We have $H(\sigma_1)=H$ if $\sigma_1 \leqslant H$, and $H(\sigma_1)=\sigma_1$ if $\sigma_1>H$.

Let E be a closed set contained in a half-plane $\sigma<\sigma_1$.

The expression '*the only possible singularities of* (A, Λ) *in* P_{σ_1} *are the points of* E' has the following meaning: if we denote by $(P_{\sigma_1}-E)$ that part of the difference $P_{\sigma_1}-E$ (the complement of $E \cap P_{\sigma_1}$ in P_{σ_1}) which forms a domain (a connected set of interior points) containing a half-plane P_{σ_2} with $\sigma_2>\sigma_c$, then $f(s)$ is holomorphic in $(P_{\sigma_1}-E)$ (and in P_{σ_2} we have: $f(s)=(A, \Lambda)$).

It is important to emphasize that $\sigma_1'<\sigma_1$ does not imply $S_{\sigma_1} \subset S_{\sigma_1'}$. This is the reason why in the definition of S_{σ_1} we use the expression '*relative to* P_{σ_1}' and not '*in* P_{σ_1}'.

Let $f(s)=(A, \Lambda)$ be uniform in P_{σ_1}; for $\varepsilon>0$ we set:

$$S_{\sigma_1}(\varepsilon) = \bigcup_{s\in S_{\sigma_1}} C(s, \varepsilon).$$

If to every $\varepsilon>0$ there corresponds a constant K $(=K(\varepsilon, \sigma_1))$ such that[2] on $\Delta(\sigma_1, \varepsilon)=\overline{(P_{\sigma_1}-S_{\sigma_1}(\varepsilon))}$: $|f(s)|<K$, we say that f is *bounded in* P_{σ_1}, *except at the singularities*. It is clear that a Taylor-D series, uniform in $\sigma>\sigma'$ is bounded except at the singularities. But this is not the case for all the Dirichlet series. For example, the function $\zeta(s)$ is not bounded, except at the singularities in any half-plane P_{σ_1} with $\sigma_1=1-\delta$ $(\delta>0)$.

Let us now suppose that $f(s)=(A, \Lambda)$, uniform in P_{σ_1}, is such that there exists a non-negative constant m with the property that to every $\varepsilon>0$ there corresponds a $K(\varepsilon, m)$ $(=K(\varepsilon, m, \sigma_1))$ such that $|f(s)|<K(\varepsilon, m)|t|^m$ holds in $\Delta(\sigma_1, \varepsilon)$ for sufficiently large $|t|$. Let μ be the greatest lower bound

of these quantities m. In other words, to every $\delta > 0$ and every $\varepsilon > 0$ there corresponds a constant

$$N(\varepsilon, \delta) \quad (= N(\varepsilon, \delta, \sigma_1))$$

such that for sufficiently large $|t|$

$$|f(s)| \leqslant N(\varepsilon, \delta)|t|^{\mu+\delta},$$

in $\Delta(\sigma_1, \varepsilon)$ and no quantity smaller than μ has this property. We then say that *f is of order μ in P_{σ_1} except at the singularities.*

Let $A(t)$ be a function that is positive for $t \geqslant 0$, increasing, tending to infinity. We say that $f(s) = (A, \Lambda)$ *does not grow faster than $A(t)$ in P_{σ_1} except at the singularities* if to every $\varepsilon > 0$ there corresponds a $K(\varepsilon) = K(\varepsilon, \sigma_1)$ such that in $\Delta(\sigma_1, \varepsilon)$ we have

$$|f(s)| \leqslant K(\varepsilon) A(|t|).$$

Let $Q(\sigma_1, t_0)$ be the quarter-plane given by $\sigma > \sigma_1$, $t > t_0$. Suppose that (A, Λ) has the following properties. Let Δ be a domain such that $Q(\sigma_1, t_0) \supset \Delta \supset Q(\sigma_2, t_0)$, where σ_2 is a certain quantity $\sigma_2 > \sigma_1$; $f(s)$, given by (A, Λ) in P_{σ_2}, being holomorphic and uniform in Δ. Suppose also that there exists a domain $D(\sigma_1, t_0)$ which contains all the domains Δ just described. Then we say that $f(s)$, given by (A, Λ), is *uniform in the quarter plane $Q(\sigma_1, t_0)$*. We denote by S_{σ_1, t_0} the set of points s satisfying the inequalities $\sigma \geqslant \sigma_1$, $t \geqslant t_0$ and not belonging to $D(\sigma_1, t_0)$. Then S_{σ, t_0} is called the *singular set of f relative to the quarter plane $Q(\sigma_1, t_0)$*. Next, let $S_{\sigma_1, t_0}(\varepsilon)$ be the union of the disks $C(s, \varepsilon)$ with $s \in S_{\sigma_1, t_0}$. If to every $\varepsilon > 0$ there corresponds a constant $K(\varepsilon)$ $(= K(\varepsilon, \sigma_1, t_0))$ such that in the closed domain $(\overline{Q(\sigma_1, t_0) - S_{\sigma_1, t_0}(\varepsilon)})$, which is the part of the closure of $Q(\sigma_1, t_0) - S_{\sigma_1, t_0}(\varepsilon)$ connected with $Q(\sigma_2, t_1)$ we have $|f(s)| < K(\varepsilon)$, then we say that *f is bounded in $Q(\sigma_1, t_0)$ except at the singularities.* If f is uniform and bounded in P_{σ_1} except at the singularities, then it has the same properties in $Q(\sigma_1, t_0)$ for any t_0.

We introduce the quantities

$$\sigma_1^*(t_0) = \sup_{s \in S_{\sigma_1, t_0}} \sigma, \qquad H^+(\sigma_1) = \lim_{t_0 \to \infty} \sigma_1^*(t_0),$$

It is evident that $H^+(\sigma_1)$ exists and that $H^+(\sigma_1) \geqslant \sigma_1$. We call $H^+(\sigma_1)$ the *ultimate (+)-abscissa of holomorphy of f in the half-plane P_{σ_1}*.

Since the first theorem of this chapter concerns functions $f(s)=(A, \Lambda)$ that are bounded, except at the singularities, in a half-plane or quarter-plane, it is essential to prove that there exist Dirichlet series which are not Taylor-D series (or series of the form $\sum a_n e^{-kns}$, where k is a constant), and which have these properties. Let (A, Λ) be an arbitrary series with $\sigma_a<0$, and let $T(z)=\sum c_n z^n$ be an entire function. Consider the function

$$F(s) = T\left[f(s) + \frac{1}{e^s - 1}\right] = T(\sum a_n e^{-\lambda_n s} + \sum e^{-ns})$$
$$= \sum c_m(\sum a_n e^{-\lambda_n s} + \sum e^{-ns})^m.$$

Since for $\sigma>0$ the series $\sum |c_m|(\sum |a_n| e^{-\lambda_n \sigma}+\sum e^{-n\sigma})^m$ converges, we see that $F(s)$ is represented for $\sigma>0$ by a Dirichlet series

$$F(s) = \sum d_n e^{-\nu_n s},$$

where the ν_n are of the form $\nu_n=\sum_{j\leqslant p} \alpha_j\lambda_j+\beta$, the α_j and β being non-negative integers. On the other hand, the function $f(s)+(e^s-1)^{-1}$ has in the half-plane $\sigma>\sigma_a$ only simple poles at the places $2k\pi i$ (where k is any integer). Besides, this function is bounded for $\sigma>\sigma_a+\varepsilon$ outside the disks $C(2k\pi i, \varepsilon)$.

In the same domain $F(s)$ is also bounded. But in each of these disks the values taken by $f(s)+(e^s-1)^{-1}$ cover a neighbourhood of infinity, and by Picard's theorem $F(s)$ takes in these disks all values, except possibly one, infinitely often. The points $2k\pi i$ are therefore (essential) singularities of F. Thus, this function has the properties required for every $\sigma_1>\sigma_{a,f}$.

Before we state our theorems, let us make a convention: if in a statement several Dirichlet series occur, for example $f(s)=(A, \Lambda)$, $\varphi(s)=(B, M)$, we attach to the various elements we have to introduce the letter indicating the relevant function. Thus, the abscissae of simple or absolute convergence of f will be denoted, respectively, by $\sigma_{c,f}$, $\sigma_{a,f}$; the abscissa of holomorphy by H_f; the singular set of f with respect to P_{σ_1} by $S_{\sigma_1,f}$, etc.

If the analytic continuation of a series $f(s)=(A, \Lambda)$ is uniform, we denote by S_f its singular set (in the entire plane), that is, the complement in the plane to the domain of existence of f (the total domain where (A, Λ) can be continued analytically).

In all the subsequent cases we assume that the Dirichlet series occurring in the conditions have an abscissa of absolute convergence without stating this every time.

The theorem we are about to state contains as a particular case Hadamard's theorem on Taylor series, although V.1.1 concerns much less general series than the next theorem (V.1.2).

THEOREM V.1.1. *Let* $f(s)=(A, \Lambda)$ *be uniform and bounded in* $Q(\sigma_1, t_0)$, *except at the singularities, and let* $\varphi(s)=(B, \Lambda)$ *be uniform and bounded in* P_{σ_2}, *except at the singularities. Let* $S_{\sigma_1,t_0,f}$ *be the singular set of* f *with respect to* $Q(\sigma_1, t_0)$ *and let* $S_{\sigma_2,\varphi}$ *be the singular set of* φ *with respect to* P_{σ_2}.

We set $c_n=a_nb_n\,(n\geqslant 1)$, $C\equiv\{c_n\}$. *The function* $F(s)=(C, \Lambda)$ *has the following properties:* $\sigma_{a,F}\leqslant\sigma_{a,f}+\sigma_{a,\varphi}$, F *is uniform in* P_{σ_3}, *where* $\sigma_3=H_f^+(\sigma_1)+\sigma_2$, *and the only possible singularities of* (C, Λ) *in* P_{σ_3} *are the points*[3] *of* $S_{\sigma_1,t_0,f}+S_{\sigma_2,\varphi}$ *and the limit points of this set.*

Let $c>\sigma_{a,f}$, $t_1>t_0$, and let us write for $z=x+iy$

$$F_T(z) = \frac{1}{Ti}\int_{c+t_1i}^{c+Ti} f(s)\varphi(z-s)\,ds,$$

where the integration is taken over the line $\sigma=c$. Then $F_T(z)$ is holomorphic for $x>c+\sigma_{a,\varphi}$, for when $\sigma=c$, we have $z-s\in P_{\sigma_{a},\varphi}$. If z satisfies this condition, we have

$$\begin{aligned} F_T(z) &= \frac{1}{Ti}\int_{c+t_1i}^{c+Ti} f(s)\sum b_n\,e^{-\lambda_n(z-s)}\,ds \\ &= \sum b_n\,e^{-\lambda_n z}\frac{1}{Ti}\int_{c+t_1i}^{c+Ti} f(s)\,e^{\lambda_n s}\,ds. \end{aligned}$$

But according to Theorem I.3.1 we have

$$\text{(V.1)}\qquad \frac{1}{Ti}\int_{c+t_1i}^{c+Ti} f(s)\,e^{\lambda_n s}\,ds = a_n + \varepsilon_n(T),$$

with lim $\varepsilon_n(T)=0$ as $(T\to\infty)$. Thus,

$$F_T(z) = \sum a_nb_n\,e^{-\lambda_n z} + \sum \varepsilon_n(T)^{-\lambda_n z}.$$

The series (C, Λ) converges absolutely for $x>\sigma_{a,\varphi}+c$, because

$$\begin{aligned} \sum_{n>p}|a_nb_n|\,e^{-\lambda_n x} &= \sum_{n>p}|b_n|\,e^{-\lambda_n x}\left|\lim_{T\to\infty}\frac{1}{Ti}\int_{c+t_1i}^{c+Ti} f(s)\,e^{\lambda_n s}\,ds\right| \\ &\leqslant \mathscr{M}(c)\sum_{n>p}|b_n|\,e^{-\lambda_n(x-c)} \end{aligned}$$

where

$$\mathscr{M}(c) \leqslant \sup |f(c + it)| \leqslant \sum |a_n| e^{-\lambda_n c}.$$

We also have $\sum |b_n| e^{-\lambda_n(x-c)} < \infty$ for $x - c > \sigma_{a,\varphi}$.

Hence the series (C, Λ) converges absolutely for $\sigma > \sigma_{a,f} + \sigma_{a,\varphi}$, and $\sigma_F \leqslant \sigma_{a,f} + \sigma_{a,\varphi}$.

For $x > \sigma_{a,\varphi} + c$ we have:

$$|F_T(z) - F(z)| = |\sum b_n e^{-\lambda_n z}[a_n + \varepsilon_n(T)] - \sum a_n b_n e^{-\lambda_n z}|$$

and since by (V.1) $|\varepsilon_n(T)| \leqslant 2\mathscr{M}(c) e^{\lambda_n c}$, we have for each p:

$$|F_T(z) - F(z)| \leqslant \sum_{n \leqslant p} |b_n \varepsilon_n(T)| e^{-\lambda_n x} + 2\mathscr{M}(c) \sum_{n > p} |b_n| e^{-\lambda_n(x-c)};$$

but on the one hand, for fixed p, $\varepsilon_n(T) \to 0$ as $T \to \infty$, $n = 1, 2, \ldots, p$, and on the other hand, $\sum_{n>p} |b_n| e^{-(\lambda_n - c)} \to 0$ as $p \to 0$, $x - c > \sigma_{a,\varphi}$. So we see that $\lim F_T(z) = F(z)$ for $x > \sigma_{a,\varphi} + c$.

Suppose now that $c > \max(\sigma_{a,f}, \sigma_1)$ and let us divide the strip $\sigma_1 \leqslant \sigma \leqslant c$ into squares whose sides are parallel to the axes and of length $\varepsilon = (c - \sigma_1)/q$ (where q is an integer). We denote by A a square of this division having the following property: the closure of the union of this square with all its adjacent squares does not contain any point of $S_{\sigma_1, t_0, f}$. Let $\mathfrak{D}(\varepsilon)$ be the set consisting of the union of all the squares A of this division. If ε is sufficiently small (q sufficiently large), then all the squares based on $\sigma = c$ form part of $\mathfrak{D}(\varepsilon)$. We assume ε to be chosen this way. Let $\mathfrak{D}^*(\varepsilon)$ be the largest domain forming part of $\mathfrak{D}(\varepsilon)$ and having $\sigma = c$ as part of its boundary. Let $\mathfrak{D}^*(\varepsilon, t_1)$ be the part of $\mathfrak{D}^*(\varepsilon)$ for which $t > t_1$. Then $f(s)$ is holomorphic and bounded in $\overline{\mathfrak{D}^*(\varepsilon, t_1)}$. Let $C(\varepsilon)$ be the boundary of $\mathfrak{D}^*(\varepsilon, t_1)$. Let $\{\alpha_m\}$ be a sequence tending to $+\infty$, with $\alpha_m > t_1$ $(m \geqslant 1)$; we denote by $\mathfrak{D}^*(\varepsilon, t_1, m)$ the part of $\mathfrak{D}^*(\varepsilon, t_1)$ for which $t < \alpha_m$, and by $C^{\cdot}(\varepsilon, t_1, m)$ the part of its boundary that does not contain the points of $\sigma = c$. It follows from Cauchy's theorem that for $x > \sigma_{a,\varphi} + c$

$$F_{\alpha_m}(z) = \frac{1}{i\alpha_m} \int_{C^{\cdot}(\varepsilon, t_1, m)} f(s)\varphi(z - s)\, ds \tag{V.2}$$

(the integral being taken in a convenient sense) and consequently $F(z)$ for the same values of x is the limit of the integral (V.2) as m tends to infinity.

Let l_m be the set of points forming the segments of $C^{\cdot}(\varepsilon, t_1, m)$ for which $t=\alpha_m$, and let L be the set of points forming the segments of $C^{\cdot}(\varepsilon, t_1, m)$ for which $t=t_1$. Since for given m the total length of each of these sets of segments does not exceed $c-\sigma_1$, and since on these segments $|f(s)|$ and $|\varphi(z-s)|$ (for $x>\sigma_{a,\varphi}+c$) are bounded, we see that when we denote by $C(\varepsilon, t_1, m)$ the set

$$C^{\cdot}(\varepsilon, t_1, m) - l_m - L$$

we have $F(z)=\lim F^{\cdot}_{\alpha_m}(z)$, where

$$F^{\cdot}_{\alpha_m}(z) = \frac{1}{i\alpha_m}\int_{C(\varepsilon,t_1,m)} f(s)\varphi(z-s)\,ds.$$

Now let Δ be a bounded domain of the plane $z=x+iy$ containing points with $x>\sigma_{a,\varphi}+c$ such that for every $z\in\Delta$

$$x > H_f^+(\sigma_1) + \sigma_2 - 3\varepsilon$$

and that for every couple z, γ with $z\in\Delta$, $\gamma\in S_{\sigma_1,t_0,f}+S_{\sigma_2,f}$ we have $|z-\gamma|>6\varepsilon$. If t_1 is sufficiently large, we have for every $s\in C(\varepsilon, t_1, m)$, $z\in\Delta$, $\beta\in S_{\sigma_2,\varphi}$

$$\Re(z-s) = x-\sigma > \sigma_2, \quad |z-s-\beta| > \varepsilon.$$

Indeed, if t_1 is sufficiently large, then for every $s\in C(\varepsilon, t_1, m)$ we have $\sigma<H_f^+(\sigma_1)+3\varepsilon$; and for $z\in\Delta$ we have $x-\sigma>\sigma_2$. If we had $|z'-s'-\beta'|<\varepsilon$ for a $z'\in\Delta$, an $s'\in C(\varepsilon, t_1, m)$, and a $\beta'\in S_{\sigma_2,\varphi}$, because to every s' indicated there corresponds an $\alpha'\in S_{\sigma_1,t_0,f}$ such that $|s'-\alpha'|<2\sqrt{2}\,\varepsilon$, we would have $|z'-\alpha'-\beta'|\leqslant|z'-s'-\beta'|+|s'-\alpha'|<(2\sqrt{2}+1)\varepsilon$, contrary to the assumption that $|z'-\alpha'-\beta'|>6\varepsilon$. With this choice of t_1, for every $s\in C(\varepsilon, t_1, m)$, $z\in\Delta$, the variable $u=z-s$ is situated in that part of the half-plane $\sigma>\sigma_2$ which is connected with the half-plane of absolute convergence of φ and in which $\varphi(s)$ is holomorphic, uniform and bounded. Thus, the functions $F^{\cdot}_{\alpha_m}(z)$ defined above form a bounded family of holomorphic functions in Δ. But in a closed part of Δ situated in the half-plane $x>\sigma_{a,\varphi}+c$ the sequence $F^{\cdot}_{\alpha_m}(z)$ converges uniformly to $F(z)=\sum a_n b_n e^{-\lambda_n s}$, and by the Stieltjes-Vitali theorem, $F^{\cdot}_{\alpha_m}(z)$ converges uniformly to a function holomorphic on every compact set in Δ. This limit is the analytic continuation of $F(z)$. Thus, the 'composite' sum $\sum a_n\, b_n\, e^{-\lambda_n s}$ can be continued analytically into every domain Δ of a kind we have defined. This proves the theorem.

Theorem V.1.1 contains as a particular case Hadamard's theorem, because for the Taylor-D series σ_1 and σ_2 can be chosen arbitrarily, at least if the functions are uniform. (For multiform functions a statement valid for Dirichlet series of an order of generality comparable to that of Theorem V.1.1 can also be obtained; later we shall have to say something about this.)

The fact that t_0 is arbitrary in Theorem V.1.1 does not improve the corresponding statement for the Taylor-D series, because f and φ are then periodic functions of period $2\pi i$. But the arbitrary choice of t_0 plays an important role in some applications of Theorem V.1.1 to general Dirichlet series. Note that for two Taylor-D series the sum $S_{\sigma_1,f}+S_{\sigma_2,\varphi}$ is closed, because the sets $S_{\sigma_1,f}$, and $S_{\sigma_2,\varphi}$ are themselves periodic of period $2\pi i$.

V.2. Composition of functions of slow growth

Before dealing with Dirichlet series of a much more general character we have to introduce some new definitions. Let $M=\{\mu_n\}$ be a strictly increasing sequence of positive numbers tending to infinity; as before, let $f(s)=(A, \Lambda)=\sum a_n e^{-\lambda_n s}$, and let k be a positive integer. Let n_0 be the smallest integer n such that $\mu_n>\lambda_1$, and let us set

$$a_n^{(k)} = 0 \quad (0 < n < n_0, \quad \text{for} \quad n_0 > 1)$$
$$a_n^{(k)} = \sum_{\lambda_m < \mu_n} (\mu_n - \lambda_m)^k a_m \quad (n \geqslant n_0).$$

The quantity $a_n^{(k)}$ is called the n-th *coefficient of f (or of* (A, Λ)*) relative to M, of order k.*

THEOREM V.2.1. *Let* $f(s)=(A, \Lambda)$ *be uniform and of order* ν *in* P_{σ_1}, *except at the singularities, and let* $\varphi(s)=(B, M)$ *be uniform and of order* μ *in* P_{σ_2}, *except at the singularities.*

Let $S_{\sigma_1,f}$ *be the singular set of f with respect to the half-plane* P_{σ_1}, *and* $S_{\sigma_2,\varphi}$ *the singular set of* φ *with respect to the half-plane* P_{σ_2}. *Let* $H_f(\sigma_1)$ *be the abscissa of holomorphy of f in* P_{σ_1}.

Let k be an integer such that $k>\nu+\mu$. *Then the series* $\sum a_n^{(k)} b_n e^{-\mu_n s}$, *where* $a_n^{(k)}$ *is the n-th coefficient of f relative to* $\{\mu_n\}$, *of order k, has an abscissa of absolute convergence not exceeding*

$$\max(\sigma_{a,\varphi}, \sigma_{a,\varphi} + \sigma_{a,f});$$

the function $F_k(s)$ represented by this series is uniform in the half-plane $\sigma > \max(H_f(\sigma_1), 0) + \sigma_2$, and the only possible singularities of $F_k(s)$ in this half-plane are the points belonging to $S_{\sigma_1,f} + S_{\sigma_2,\varphi}$, the limit points of this set, and the points of $S_{\sigma_2,f}$.

Let $c > \max(0, \sigma_{a,f})$ and consider

$$\text{(V.3)} \qquad F(z) = \frac{k!}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{f(s)\varphi(z-s)}{s^{k+1}}\, ds,$$

the integration being taken over the line $\sigma = c$.

$F(z)$ is holomorphic if $x > c + \sigma_{a,\varphi}$ $(z = x + iy)$, for when $\sigma = c$, the variable $z - s$ remains in the half-plane of absolute convergence of φ. For such values of z we can write:

$$\frac{k!}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s) \sum b_n\, e^{-\mu_n(z-s)} \frac{ds}{s^{k+1}}$$
$$= \sum_n b_n\, e^{-\mu_n z} \frac{k!}{2\pi i} \int_{c-i\infty}^{c+i\infty} \sum_m a_m\, e^{(\mu_n - \lambda_m)s} \frac{ds}{s^{k+1}}$$

and I.3.3 enables us to write (for these values of z)

$$F(z) = \sum_{n \geqslant n_0} b_n\, e^{-\mu_n z} \sum_{\lambda_m < \mu_n} a_m(\mu_n - \lambda_m)^k = \sum a_n^{(k)} b_n\, e^{-\mu_n z},$$

where n_0 is the smallest integer n such that there exists a λ_m less than μ_n.

The last series for $F(z)$ converges for $x > c + \sigma_{a,\varphi}$. But (V.3) is still valid when we replace $f(s)$ by $\sum |a_n|\, e^{-\lambda_n s}$ and $\varphi(s)$ by $\sum |b_n|\, e^{-\mu_n s}$.

Consequently the series

$$\sum_{n \geqslant n_0} \sum_{\lambda_m < \mu_n} |a_m|(\mu_n - \lambda_m)^k |b_n|\, e^{-\mu_n x}$$

converges for $x > c + \sigma_{a,\varphi}$, which proves that the abscissa of absolute convergence of $\sum a_n^{(k)} b_n\, e^{-\mu_n s}$ does not exceed $\max(\sigma_{a,\varphi}, \sigma_{a,f} + \sigma_{a,\varphi})$ (c is only subject to the inequality $c > \max(0, \sigma_{a,f})$).

Evidently $F(z) = F_k(z)$.

Now we choose $c > \max(0, \sigma_1, \sigma_{a,f})$; and as in the proof of Theorem V.1.1 we divide the strip $\sigma_1 \leqslant \sigma \leqslant c$ into squares of sides parallel to the axes and of length $\varepsilon = (c - \sigma_1)/q$; let $\mathfrak{D}_1(\varepsilon)$ be the set consisting of the union of the squares of this division having the following property: the closure of the union of such a square with all its adjacent squares does not contain any point of $S_{\sigma_1,f}$ nor the point $s = 0$. If ε is sufficiently small,

all the squares based on $\sigma=c$ form part of $\mathfrak{D}_1^*(\varepsilon)$, the largest domain contained in $\mathfrak{D}_1(\varepsilon)$ having $\sigma=c$ as part of its boundary. The function f is holomorphic in $\overline{\mathfrak{D}_1^*(\varepsilon)}$, and to every $\delta_1>0$ there corresponds a constant $N_1(\delta_1)$ $(=N_1(\delta_1, \varepsilon, \sigma_1))$ such that $|f(s)|<N_1(\delta_1)|t|^{\mu+\delta_1}$ on $\overline{\mathfrak{D}_1^*(\varepsilon)}$ for sufficiently large $|t|$.

Let $\mathfrak{D}_1^*(\varepsilon, T)$ be the part of $\mathfrak{D}_1^*(\varepsilon)$ for which $|t|<T$, and let $C_1^*(\varepsilon, T)$ be the part of its boundary that does not contain the points of $\sigma=c$. For $x>\sigma_{a,\varphi}+c$ we have, by Cauchy's theorem,

$$\int_{C_1^*} f(s)\varphi(z-s)\frac{ds}{s^{k+1}} = \int_{c-iT}^{c+iT} f(s)\varphi(z-s)\frac{ds}{s^{k+1}}$$

where the first integral is taken in a convenient sense over $C_1^*=C_1^*(\varepsilon, T)$, and the second over the segment of the line $\sigma=c$.

On all parts of $C_1^*(\varepsilon, T)$ on which $t=-T$ or $t=T$, the integrals tend to 0 as T tends to $+\infty$. It follows that

$$\text{(V.4)} \qquad F_k(z) = \frac{k!}{2\pi i}\int_{C_1^*(\varepsilon)} f(s)\varphi(z-s)\frac{ds}{s^{k+1}},$$

where $C_1^*(\varepsilon)$ is the part of the boundary of $\mathfrak{D}_1^*(\varepsilon)$ that does not contain the points of $\sigma=c$.

Let Δ be a bounded domain of the plane of $z=x+iy$ containing points $x>\sigma_{a,\varphi}+c$ and having the following properties: $x>\max(H_f(\sigma_1), 0)+\sigma_2+2\varepsilon$ for every $z\in\Delta$, and $|z-\gamma|>6\varepsilon$ for every $z\in\Delta$ and $\gamma\in(S_{\sigma_2,\varphi}+S_{\sigma_1,f})\cup S_{\sigma_2,\varphi}$. We see that

$$\Re(z-s) = x-\sigma > \sigma_2,$$

that $|z-s-\beta|>\varepsilon$ for every $z\in\Delta$, every $s\in C_1^*(\varepsilon)$ and every $\beta\in S_{\sigma_2,\varphi}$. This can be shown in a manner similar to that we have used to demonstrate an analogous fact in the proof of Theorem V.1.1. Consequently, if $s\in C_1^*(\varepsilon)$, $z\in\Delta$, then the variable $u=z-s$ is situated in that part of the half-plane $\sigma>\sigma_2$ which is connected with the half-plane of absolute convergence where $\varphi(s)$ is holomorphic and uniform and where to every $\delta_2>0$ there corresponds a constant $N_2(\delta_2)$ $(=N_2(\delta_2, \varepsilon, \sigma_2))$ with the property:

$$|\varphi(s)| \leqslant N_2(\delta_2)|t|^{\nu+\delta_2}$$

for sufficiently large $|t|$. Since Δ is bounded, we see that

$$|\varphi(z-s)| < N_3(\delta_2)|t|^{\nu+\delta_2}$$

for sufficiently large $|t|$ if $s \in C_1^*(\varepsilon)$, $z \in \Delta$. Since $\nu+\mu<k$, clearly δ_1 and δ_2 can be chosen so that

$$\mu + \nu + \delta_1 + \delta_2 < k,$$

and the integral (V.4) converges uniformly in z on every compact set in Δ. Hence $F_k(z)$ is holomorphic in Δ, and the theorem is proved. Theorem V.2.1 also contains Hadamard's theorem, at least if we assume that the Taylor series represents a uniform function. In fact, let:

$$\psi(s) = \sum d_n\, e^{-ns}, \qquad \varphi(s) = \sum b_n\, e^{-ns}$$

(to simplify the calculations we assume that $d_1=0$; also $d_0=b_0=0$ as everywhere in this text) and let $z_0=e^{-s_0}$ be one of the singularities of $\sum d_n z^n$ on its circle of convergence. The function

$$\psi_1(s) = \sum d_n\, e^{-ns_0}\, e^{-ns} = \sum A_n\, e^{-ns}$$

has a singularity at the point $s=0$. We set

$$(1 - s^{-s})^2\, e^s \psi_1(s) = f(s) = \sum a_n\, e^{-ns}.$$

Then

$$\begin{aligned} A_n &= (n-1)a_1 + (n-2)a_2 + \cdots + a_{n-1} \\ &= \sum_{m<n} (n-m)a_m \quad (n \geqslant 2). \end{aligned}$$

The singularities of f are those of ψ_1, except the points $2k\pi i$, which may be singular for ψ_1 but not for f. Since for the Taylor-D series f and φ the quantities σ_1 and σ_2 can be taken negative, of arbitrarily large absolute value, and since for such a choice of σ_1 and σ_2 we can always take k to be equal to 1, we see by Theorem V.2.1 that the only singularities of $F(s)=\sum a_n^{(1)} b_n\, e^{-ns}$, where

$$a_n^{(1)} = \sum_{m<n} a_m(n-m) = A_n \quad (n > 1),$$

and $a_n^{(1)}=0$, are the points of S_f+S_φ and of S_φ.

Let S_{ψ_1} be the set of singularities of ψ_1; since S_{ψ_1} contains S_f and the point $s=0$, we see that $S_{\psi_1}+S_\varphi=(S_f+S_\varphi) \cup S_\varphi$. In other words, the only possible singularities of

$$F(s) = \sum A_n b_n\, e^{-ns} = \sum d_n b_n\, e^{-ns_0}\, e^{-ns}$$

are the points of $S_{\psi_1}+S_\varphi$. Clearly, every point of S_ψ, the set of singularities of ψ, is obtained by adding s_0 to a point of S_{ψ_1}. In other words, the only possible singularities of $\Phi=\sum d_n b_n e^{-ns}$ are the points of $S_\psi+S_\varphi$. This is the most precise statement of Hadamard's theorem, at least for uniform functions. For multiform functions we shall make the appropriate remarks later.

Coming back to Theorem V.2.1, we see that if $\sigma_1<0$ and if $s=0$ does not belong to $S_{\sigma_1,f}$, then for sufficiently small $\varepsilon>0$ the boundary $C_1^*(\varepsilon)$ contains a square (of side 3ε) containing in its interior the point $s=0$, the square being separated from the other points of $C_1^*(\varepsilon)$. Let $c(\varepsilon)$ be the boundary of this square. The part of $F_k(z)$ obtained by integration over $c(\varepsilon)$ gives (by the residue theorem) for sufficiently large x:

$$\begin{aligned} I_k(z) &= \frac{k!}{2\pi i}\int_{c(\varepsilon)} f(s)\varphi(z-s)\,\frac{ds}{s^{k+1}} \\ &= \frac{\partial^k[f(s)\varphi(z-s)]}{\partial s^k} \quad (s=0) \\ &= (-1)^k[f(0)\varphi^{(k)}(z) - \tbinom{k}{1}f'(0)\varphi^{(k-1)}(z) \\ &\quad + \cdots + (-1)^k f^{(k)}(0)\varphi(z)]. \end{aligned}$$

On the other hand, we see by following the proof of Theorem V.2.1 that the only possible singularities of the function

$$F_k(z) - I_k(z) = \frac{k!}{2\pi i}\int_{C_2^*(\varepsilon)} f(s)\varphi(z-s)\,\frac{ds}{s^{k+1}},$$

where $C_2^*(\varepsilon)=C_1^*(\varepsilon)-c(\varepsilon)$, are the points of $S_{\sigma_1,f}+S_{\sigma_2,\varphi}$. From the form of $I_k(z)$ it follows that the only possible singularities of this function are those of $\varphi(z)$. Theorem V.2.1 can therefore be made more precise in the following fashion:

THEOREM V.2.2. *Under the conditions of Theorem* V.2.1, *if* $s=0$ *is a regular point of* $f(s)$, *then the function* $F_k(s)=\sum a_n^{(k)} b_n e^{-\mu_n s}$ *is uniform in the half-plane* $\sigma>H_f(\sigma_1)+\sigma_2$, *and the only possible singularities in this half-plane of the function*

$$F_k(s) - I_k(s),$$

where

$$\begin{aligned} I_k(s) &= (-1)^k[f(0)\varphi^{(k)}(s) - \tbinom{k}{1}f'(0)\varphi^{(k-1)}(s) \\ &\quad + \cdots + (-1)^k f^{(k)}(0)\varphi(s)], \end{aligned}$$

are the points of $\overline{S_{\sigma_1,f}+S_{\sigma_2,\varphi}}$.

The fact that in Theorems V.2.1 or V.2.2 the two sequences $\Lambda=\{\lambda_n\}$ and $M=\{\mu_n\}$ may be chosen completely independently allows a large number of interesting applications. Thus, since the distribution of singularities of a Taylor series is much better known than that of Dirichlet series, one would achieve considerable progress in reducing the problem of distribution of singularities of a Dirichlet series to that concerning on the one hand the Taylor series whose coefficients depend on the sequences A and Λ (the coefficients and the exponents of a given series) and on the other hand, to an investigation of the singularities of a Dirichlet series whose coefficients are simple (and given in advance). For example, we shall show that in fairly frequent cases an indication of the position of the singularities of (A, Λ) can be obtained by starting out from a knowledge of the places of the singularities of $\sum d_n e^{-\lambda_n s}$, where the d_n can be expressed in a simple manner by the λ_n, and of those of a Taylor series whose coefficients are formed by means of the a_n and λ_n.

Let $\Lambda=\{\lambda_n\}$, with $\lambda_1>1$, let $C=\{c_n\}$ be a complex sequence, and k a positive integer.

We say that *C has $G=\{g_n\}$ as a generating sequence of order k with respect to Λ* if for every $n\geqslant 1$:

$$c_n = \sum_{m<\lambda_n} (\lambda_n - m)^k g_m.$$

In general, a sequence C does not have a generating sequence of a given order with respect to a given sequence Λ. But it is easy to indicate conditions on Λ for *every sequence C* to have a generating sequence of a given order k.

V.2.3. *Let e_n be the largest integer smaller than λ_n $(n\geqslant 1)$. If there exists a sequence $\{n_j\}$ such that $e_{n_j}=n_j$ and that the determinant Δ_{n_j} $(j\geqslant 1)$ of order n_j whose term in the p-th row and q-th column is $(\lambda_p-q)^k$ for $1\leqslant p\leqslant n_j$, $q\leqslant e_p$, and is 0 for $1\leqslant p<n_j$, $e_p<q\leqslant e_{n_j}$, is different from 0, then every sequence C has a generating sequence of order k with respect to Λ.*

In fact, every system S_j of equations (in the unknowns g_n)

$$\sum_{m<\lambda_1} (\lambda_1 - m)^k g_m = c_1$$

$$\vdots$$

$$\sum_{m<n_j} (\lambda_{n_j} - m)^k g_m = c_n$$

has a system of solutions. If $n_{j_1} < n_{j_2}$ ($n_{j_1}, n_{j_2} \in \{n_j\}$), then the quantities $g_1^{(j_2)}, g_2^{(j_2)}, \ldots, g_{n_{j_1}}^{(j_2)}$ which are the first n_{j_1} terms of the solution of the system (S_{j_2}) also form a system of solutions of the system (S_{j_1}). The set of solutions of all the systems (S_j) $(j \geqslant 1)$ thus constitutes a sequence $\{g_n\}$ (to every n there corresponds a single g_n) satisfying all the systems of equations

$$\begin{aligned}
&\sum_{m<\lambda_1} (\lambda_1 - m)^k g_m = c_1\\
&\quad\vdots\\
&\sum_{m<\lambda_n} (\lambda_n - m)^k g_m = c_n.
\end{aligned}$$

V.2.4. *If* $n < \lambda_n \leqslant n+1$ $(n \geqslant 1)$, *then every sequence* C *has a generating sequence of every order* k *with respect to* Λ.

Here $n_j = j$, and it is clear that

$$\Delta_n = [(\lambda_1 - 1)(\lambda_2 - 2)\ldots(\lambda_n - n)]^k \neq 0;$$

the conclusion therefore follows from V.2.3.

Theorem V.2.1 also allows us to state the following theorem.

THEOREM V.2.5. *Suppose that the function* $F(s)$ *is represented by* (A, Λ) $(\lambda_1 > 1)$ *and suppose that the function* $\varphi(s)$, *represented by a series* (D, Λ), *where* $D = \{d_n\}$ $(d_n \neq 0, n \geqslant 1)$, *is uniform of order* μ, *except at the singularities, in* P_{σ_2}. *Suppose that* $\{a_n d_n^{-1}\}$ *has a generating sequence* $\{g_n\}$ *of integral order* $k > \mu$ *with respect to* Λ, *with*

$$\limsup \frac{\log |g_n|}{n} = a < \infty,$$

the function $f(s)$ *represented by* $(G, N) = \sum g_n e^{-ns}$ *being uniform.*

Then F *is uniform and the only possible singularities in the half-plane* $\sigma > \max(a, 0) + \sigma_2$ *are the points of* $\overline{S_{\sigma_2,\varphi} + S_f}$, *and the points of* $S_{\sigma_2,\varphi}$.

This theorem is, in effect, a special case of Theorem V.2.1, where the quantities g_n play the role of the a_n in Theorem V.2.1, the d_n play the role of the b_n in the same theorem, and σ_1 is an arbitrary quantity with $H_f(\sigma_1) = a$ for $\sigma_1 < a$.

So we see, by Theorem V.2.5, that whenever Λ satisfies the conditions of V.2.3 (or, in particular, the conditions of V.2.4), then there is only one Dirichlet series that is not a Taylor-D series whose singularities are used to provide information on the singularities of the given series (A, Λ). And, what is essential, the Dirichlet series in question is a series which we can choose in advance, $(D, \Lambda) = \sum d_n e^{-\lambda_n s}$. The other series with which 'we compose' is a Taylor-D series.

It can be advantageous to choose $d_n = 1$ $(n \geqslant 1)$, in which case the series (D, Λ) becomes $\sum e^{-\lambda_n s}$, the analogue to the series $\sum e^{-ns}$ for $\Lambda = N$. One can also choose $d_n = 1/\Lambda'(\lambda_n)$, where

$$\text{(V.5)} \qquad \Lambda(z) = \prod \left(1 - \frac{z^2}{\lambda_n^2}\right) = \Lambda_0(iz),$$

and where $\Lambda_0(z)$ has been defined in I.1.2. In fact, it is known that if Λ is a sufficiently regular sequence, then the series

$$\text{(V.6)} \qquad \Phi(s) = \sum \frac{e^{-\lambda_n s}}{\Lambda'(\lambda_n)}$$

presents certain interesting and important features. For example, its analytic continuation can be expressed by an integral which represents the extension to the complex plane of the Fourier transform of $1/\Lambda_0(ix)$.

Incidentally, we find that $|\Lambda'(\lambda_n)| = 2/\lambda_n \Lambda_n$, where $\{\Lambda_n\}$ is the sequence associated with Λ. And for $\Lambda = N$ $(\lambda_n = n, n \geqslant 1)$ we have $\Lambda(z) = \sin \pi z / \pi z$,

$$\Lambda'(n) = (-1)^n / n (\Lambda_n = 2),$$
$$\Phi(s) = \sum (-1)^n n e^{-ns}.$$

We are going to indicate properties of the series (V.6). But first we have to evaluate in a particular case the Λ_n in a fairly precise form.

V.2.6. *If the step of Λ is positive and if there is an integer k such that*

$$|\lambda_n - n| \leqslant k \quad (n \geqslant 1),$$

then there exists a constant A such that

$$\Lambda_n \leqslant A \lambda_n^{4k} \quad (n \geqslant 2k+1),$$

where A depends only on $\alpha = \inf(\lambda_{n+1} - \lambda_n)$ and on k.

We may assume that $k > 0$, since we have just seen for $\Lambda = N$ that $\Lambda_n = 2$ $(n \geqslant \nu)$. So we have:

$$\Lambda_n^{-1} = \prod_{m \leqslant n-1} \left(\frac{\lambda_n^2}{\lambda_m^2} - 1\right) \prod_{m \geqslant n+1} \left(1 - \frac{\lambda_n^2}{\lambda_m^2}\right) = C_n D_n,$$

where, on the one hand, for $n > 2k+1$,

$$C_n \geqslant \prod_{m \leqslant n-2k-1} \left[\frac{\lambda_n^2}{(m+k)^2} - 1\right] \prod_{n-2k \leqslant m \leqslant n-1} \left(\frac{\lambda_n^2}{\lambda_m^2} - 1\right),$$

and on the other hand,

$$D_n \geqslant \prod_{n+1 \leqslant m \leqslant n+2k} \left(1 - \frac{\lambda_n^2}{\lambda_m^2}\right) \prod_{m \geqslant n+2k+1} \left[1 - \frac{\lambda_n^2}{(m-k)^2}\right].$$

Let p_n be the nearest integer to λ_n. Evidently,

$$n - k - 1 < p_n < n + k + 1,$$

and the preceding inequalities allow us to write

$$\begin{aligned} \Lambda_n^{-1} \geqslant & \prod_{0<|m-n| \leqslant 2k} \left|1 - \frac{\lambda_n^2}{\lambda_m^2}\right| \prod_{k+1 \leqslant m \leqslant n-k-1} \left(\frac{\lambda_n^2}{m^2} - 1\right) \\ & \times \prod_{m \geqslant n+k+1} \left(1 - \frac{\lambda_n^2}{m^2}\right) \\ & = \prod\nolimits_1 \prod\nolimits_2 \prod\nolimits_3^{-1} \end{aligned}$$

where for fixed n we have:

$$\prod\nolimits_1 = \prod_{0<|m-n| \leqslant 2k} |\lambda_m - \lambda_n| \prod (\lambda_m + \lambda_n) / \prod \lambda_m^2,$$

which proves that $\prod_1 > A_1 \lambda_n^{-4k}$, with A_1 depending only on α and k; also:[4]

$$\prod\nolimits_2 = [|\sin \pi\lambda_n| / \pi |p_n - \lambda_n|][p_n^2/\lambda_n(p_n + \lambda_n)] \geqslant A_2 > 0.$$

Finally, we have:

$$\prod\nolimits_3 = \prod_{m \leqslant k} \left|\frac{\lambda_n^2}{m^2} - 1\right| \cdot \prod_{\substack{0<|m-n| \leqslant k \\ m \neq p_n}} \left(1 - \frac{\lambda_n^2}{m^2}\right) = \prod\nolimits_4 \cdot \prod\nolimits_5,$$

with $\prod_4 \leqslant A_3 \lambda_n^{2k}$, $\prod_5 \leqslant A_4 \lambda_n^{-2k}$, the constants A_1, A_2, A_3, A_4 depending only on α and on K. From this V.2.6 follows immediately.

Here is the lemma concerning the Dirichlet series with exponents Λ which can play a role similar to that of the series $\sum e^{-ns}$, or more precisely, $\sum (-1)^n n\, e^{-ns}$ when $\Lambda = N$.

V.2.7. *If the step of* Λ *is positive and if there exists a constant* k *such that*

$$|\lambda_n - n| \leqslant k,$$

then the series

$$\Phi(s) = \sum \frac{e^{-\lambda_n s}}{\Lambda'(\lambda_n)},$$

where $\Lambda(z)$ *is defined by* (V.5), *converges absolutely for* $\sigma > 0$.

The function $\Phi(s)$ (*the analytic continuation of the series*) *is even, holomorphic in the part of the plane from which the half-lines* $\sigma = 0$, $|t| \geqslant \pi$ *have been removed; and in the strip* $|t| < \pi$ *we have*

$$\Phi(s) = \frac{-1}{2\pi} \int \frac{e^{-isy}}{\Lambda(iy)}\, dy,$$

the integral converging uniformly and absolutely for $|t| \leqslant \pi - \varepsilon$, *for every* $0 < \varepsilon < \pi$. *In the strip* $|t| < \pi - \varepsilon$ *the function* $\Phi(s)$ *is bounded.*

By V.2.6 and by the equality (see p. 69) $|\Lambda'(\lambda_n)| = 2/\lambda_n \Lambda_n$, there exist two constants M and d such that

$$|\Lambda'(\lambda_n)|^{-1} \leqslant M \lambda_n^d.$$

Since the step of Λ is positive, we see that the series (V.6) converges absolutely for $\sigma > 0$.

Now we consider in the plane $z = x + iy = r\, e^{i\theta}$ a closed contour C_R formed from the segments $\theta = a$, $\theta = -a$, $0 < a < \pi/2$, $r \leqslant R$, and the arc of the circle $r = R$, $|\theta| < a$, where R is chosen so that no λ_n is equal to R. By Cauchy's theorem we have:

$$\frac{1}{2\pi i} \oint_{C_R} \frac{e^{-sz}}{\Lambda(z)}\, dz = \sum_{n \leqslant N(R)} \frac{e^{-\lambda_n s}}{\Lambda'(\lambda_n)}$$

where $N(R)$ is the number of λ_n less than R. We show that when we set $R_m = (\lambda_m + \lambda_{m+1})/2$, there exist two constants C and c such that

$$\text{(V.7)} \qquad A_m = \min |\Lambda(R_m\, e^{i\theta})| \geqslant C R_m^{-c} \quad (m \geqslant 1).$$

Evidently

$$\text{(V.8)} \qquad A_m = \Lambda(R_m) = \prod \left| 1 - \frac{R_m^2}{\lambda_n^2} \right|,$$

and if we denote by $M=\{\mu_n\}$ the increasing sequence formed by the elements of Λ and the quantity R_m (for given m), then we have

$$|\Lambda(R_m)|^{-1} = M_{m+1},$$

where $\{M_n\}$ is the sequence associated with M. Now it follows from the conditions and the way in which M has been formed that there exists a positive integer l such that:

$$|\mu_n - n| \leqslant l,$$

the step of M being positive; consequently, by V.2.6 we have

$$M_{m+1} = |\Lambda(R_m)|^{-1} \leqslant C^{-1}\mu_{m+1}^{4l} = C^{-1}R_m^{4l}$$

which gives (V.7) (when V.8 is taken into account).

It follows from (V.7) that on the arc $r=R_m$, $|\theta|\leqslant a<\pi/2$, we have for every $\varepsilon>0$ and $m\geqslant m(\varepsilon)$:

$$|\Lambda(z)| \geqslant e^{-\varepsilon r},$$

and for real positive s we can write, as long as $r=R_m$, $|\theta|<a$ $(m>m(\varepsilon))$,

$$\left|\frac{e^{-sz}}{\Lambda(z)}\right| \leqslant e^{-(s\cos a-\varepsilon)r}.$$

So we see that for real positive r (the integral being taken in the positive sense)

$$\text{(V.9)}\qquad \lim \frac{1}{2\pi i}\oint_{C_{R_m}} \frac{e^{-sz}}{\Lambda(z)}\,dz$$

$$= \lim \frac{1}{2\pi i}\left[-\int_0^{R_m} \frac{e^{-sre^{ia}}}{\Lambda(re^{ia})}\,e^{ia}\,dr + \int_0^{R_m} \frac{e^{-sre^{-ia}}}{\Lambda(re^{-ia})}\,e^{-ia}\,dr\right]$$

$$= \sum \frac{e^{-\lambda_n s}}{\Lambda'(\lambda_n)}.$$

From the conditions on Λ it follows easily that for $a\leqslant\theta\leqslant\pi/2$ we have for every $\delta>0$

$$\left|1-\frac{z^2}{(n-\frac{1}{2})^2}\right| \leqslant \left|1-\frac{r^2}{\lambda_n^2}\right|\left(1+\frac{\delta^2 r^2}{4n^2}\right),$$

provided that r and n are sufficiently large. By using next the canonical product for $\cos\pi z$ we can state that

$$|\cos\pi z|\,|\Lambda(z)|^{-1} \leqslant C_1\prod\left(1+\frac{\delta^2r^2}{4n^2}\right) = C_2\left|\sin\left(\frac{\pi i\delta r}{2}\right)\right|\frac{1}{\delta r},$$

where C_1 and C_2 are constants, which proves that for sufficiently large r and $a \leqslant |\theta| \leqslant \pi/2$:

(V.10) $\quad |\cos \pi z| < |\Lambda(z)|\, e^{\delta r}.$

It is now easy to see that for $a \leqslant |\theta| \leqslant \pi/2$ the integral

$$\text{(V.11)} \quad I_\theta(s) = \frac{1}{2\pi i} \int_0^\infty \frac{e^{-sre^{i\theta}}}{\Lambda(r\,e^{i\theta})}\, e^{i\theta}\, dr$$

converges absolutely and uniformly in the part of the plane of $s = \sigma + it$ defined by

$$t \sin\theta - \sigma\cos\theta \leqslant \pi|\sin\theta| - 2\delta.$$

Consequently, the integral (V.11) represents in this half-plane a holomorphic function. It is sufficient to observe that for given $\delta > 0$ by (V.10) we have

$$\text{(V.12)} \quad |e^{-sre^{i\theta}}| \leqslant L|\Lambda(r\,e^{i\theta})|\, e^{t\sin\theta - \sigma\cos\theta + \delta|\sin\theta|},$$

where L is a constant.

Coming back to (V.9) we see that each of the integrals in the expression

$$\frac{1}{2\pi i}\int \frac{e^{-sr\,e^{ia}}}{\Lambda(r\,e^{ia})}\, e^{ia}\, dr - \frac{1}{2\pi i}\int \frac{e^{-sr\,e^{ia}}}{\Lambda(r\,e^{-ia})}\, e^{-ia}\, dr$$

converges absolutely and uniformly in every part of the plane defined by

(V.13) $\quad |t| \leqslant \sigma\cot a + \pi - \varepsilon,$

and that its sum is $-\Phi(s)$.

By applying Cauchy's theorem and taking (V.12) into account, we see that for real positive s

$$I_a(s) = I_{-a}(s) = I_{\pi/2}(s) = I_{-\pi/2}(s) = I_0(s) = I_\pi(s).$$

The integral in our statement, which converges absolutely for $|t| < \pi$ (by (V.13) with $a = \pi/2$), converges uniformly in every strip $|t| < \pi - \varepsilon$ and represents $-\Phi(s)$ in it; this function is bounded in it. Since $\Lambda(z)$ is an even function, so is $\Phi(s)$, and this completes the proof of our proposition.

Remark. If the sequence Λ is such that to every positive integer k there corresponds at most one λ_n for which $k < \lambda_n \leqslant k+1$, in other words, if in the notation of V.2.3 $e_{n+1}-e_n \geqslant 1$ $(n \geqslant 1)$ (with $\lambda_1 > 1$), then the series (A, Λ) can be written in the form $\sum a'_p\, e^{-L_p s}$, where $L_{e_n}=\lambda_n$ and L_p, for p different from all the e_n, can be chosen arbitrarily in $]p, p+1]$ with $a'_{e_n}=a_n$ and $a'_p=0$ when p is distinct from all e_n. Consequently, by Theorem V.2.5 the study of the distribution of the singularities of (A, Λ) can be reduced to that of the singularities of a function represented by a Dirichlet series (D, L), $L \equiv \{L_n\}$, where the coefficients $D \equiv \{d_n\}$ are chosen arbitrarily, and to that of the distribution of singularities of a Taylor-D series $\sum g_n\, e^{-ns}$, where G is a generating sequence of a certain order with respect to the sequence $C=\{c_n\}$, with $c_n=a_n d_n^{-1}$.

The order of the generating sequence depends, of course, on that of (C, L), except at the singularities (if such an order exists).

In any case, it seems essential to us to emphasize that in frequent and important cases one can 'separate' from each other the difficulties of the problem of studying the singularities of a Dirichlet series which stem from the double presence of the sequence A and the sequence Λ, by studying, on the one hand, the series in which only Λ takes part (for example, $\sum e^{-\lambda_n s}$ or $\sum [\Lambda'(\lambda_n)]^{-1}\, e^{-\lambda_n s}$), and on the other hand, by investigation of the singularities of a Taylor series.

Some of the theorems we have established can also serve in the opposite direction by permitting us to state that a function represented by a Dirichlet series of general type necessarily has singularities at given places.

Let $f(s)$ be a function represented by (A, Λ) and let k be a positive integer. The n-th coefficient of f with respect to the sequence N, of order k, is called the n-th *Taylor coefficient of f, of order k*. We denote this quantity by $a_T^{(k)}(n)$.

THEOREM V.2.8. *Let $f(s)=(A, \Lambda)$ be uniform in P_{σ_1}, and suppose that f is of order ν, except at the singularities, in this half-plane.*

Let β be a singularity of the function $F(s)$ represented by $\sum a_T^{(k)}(n)\, e^{-ns}$, where k is an integer with $k > \nu$, and suppose that β is not of the form $2m\pi i$, where m is an integer. To every $\varepsilon > 0$ there corresponds an integer p such that there exists a point α belonging to $S_{\sigma_1, f}$ for which

$$|\alpha - (2p\pi i + \beta)| < \varepsilon.$$

The function $\varphi(s)=\sum e^{-ns}$ is uniform in every P_{σ_2} and is there bounded, except at the singularities. We can therefore apply Theorem V.2.1 to f

and φ with $k>\nu$. The only possible singularities of F in the entire plane are consequently the points of $\overline{S_{\sigma_1,f}+S_\varphi}=\bar{S}$ and the points of S_φ. But the points of S_φ are at the places $2m\pi i$. If β is a singularity of F, but not of the form $2m\pi i$, that is, if it does not belong to S_φ, then this point must necessarily belong to $\bar{S}$. In other words, either there exists a point $\alpha_1 \in S_{\sigma_1,f}$ and an integer m such that $\beta=\alpha_1+2m\pi i$, or there exists a sequence α_n with $\alpha_n \in S_{\sigma_1,f}$ $(n\geqslant 1)$ and a sequence of integers $\{m_n\}$ such that $\beta=\lim(\alpha_n+2m_n\pi i)$. In the latter case, for given $\varepsilon>0$ and sufficiently large n

$$|\beta-\alpha_n-2m_n\pi i|<\varepsilon.$$

Thus, the theorem is proved.

The theorem we have just proved enables us to associate with every theorem that provides a condition for a given point to be the place of a singularity of a Taylor series a theorem on general Dirichlet series (of finite order, except at the singularities) indicating the position of some of its singularities.

V.3. 'Fictitious composition' of singularities

It is important to be able to state theorems on the composition of Dirichlet series for an arbitrary Λ, without reference to the order of the functions (except at the singularities) and, *a fortiori*, without having to assume the functions to be bounded (of course, except at the singularities). One can make such statements by adding to the sets $\overline{S_f+S_\varphi}$ and S_φ other sets that are included in the preceding ones when $\Lambda=M=N$. Before we come to such a statement, we have to make some new definitions and some preliminary remarks.

Suppose that $f(s)=(A,\Lambda)$ is uniform in P_{σ_1}. We denote by $E_f(a,\sigma_1)$ the set of points in P_{σ_1} where $f(s)=a$. Let $\varphi(s)$ be uniform in P_{σ_2}.

We define a set $P=P[f,\sigma_1;\varphi,\sigma_2]$ in the following manner: $\gamma\in P$ if and only if to every value of a except at most one and to every $\varepsilon>0$ there corresponds a $\rho_a=\rho_a(\varepsilon)\in E_f(a,\sigma_1)$ and a $\beta=\beta(\varepsilon)\in S_{\sigma_2,\varphi}$ such that $|\gamma-\rho_a-\beta|<\varepsilon$. If $\Lambda=M=N$ (that is, if the two series are Taylor-D series), then $P[f,\sigma_1;\varphi,\sigma_2]$ is a subset of $S_{\sigma_1,f}+S_{\sigma_2,\varphi}$:

$$P[f,\sigma_1;\varphi,\sigma_2]\subset S_{\sigma_1,f}+S_{\sigma_2,\varphi}.$$

In fact, let $\beta_n\in S_{\sigma_2,\varphi}$, and $\rho_{\omega_n}\in E_f(\omega_n,\sigma_1)$ be such that $\lim\omega_n=\infty$, $\lim(\rho_{\omega_n}+\beta_n)=\gamma$, which is possible with the given definition of γ. But since in our case ($\Lambda = M = N$) the functions f and φ are periodic of period $2\pi i$, by writing $\rho_{\omega_n} = \rho'_{\omega_n} + i\rho''_{\omega_n}$ we may assume that

$$0 \leqslant \rho''_{\omega_n} < 2\pi$$

and that there exist two integers k_1 and k_2 such that by setting $\beta_n = \beta'_n + i\beta''_n$ we have $2k_1\pi\beta \leqslant \beta''_n \leqslant 2k_2\pi$, where all the equalities and inequalities are valid for every $n \geqslant 1$.

On the other hand, let $\sigma' > \sigma_{a,f}$. If $|\omega_n| > \mathrm{Sup}_{\sigma \geqslant \sigma'} |f(s)|$ for $n > n_0$, then all the corresponding ρ_{ω_n} $(n > n_0)$ are situated in a rectangle Δ_1 defined by $\sigma_1 \leqslant \sigma \leqslant \sigma'$, $0 \leqslant t \leqslant 2\pi$; and consequently the β_n $(n > n_0)$ are likewise situated in a rectangle Δ_2. Hence there exists a sequence $\{n_j\}$, *a* $\rho \in \Delta_1$ and a $\beta \in \Delta_2$ such that $\lim \rho_{\omega_{n_j}} = \rho$, $\lim \beta_{n_j} = \beta$, $\gamma = \rho + \beta$. But then $\rho \in S_{\sigma_2, f}$, $\beta \in S_{\sigma_2, \varphi}$, which proves our proposition.

It is equally easy to see by Picard's theorem that if $\alpha_1 \in S_{\sigma_1, f}$ and if α_1 is an essential singularity of f, then $\alpha_1 + \beta \in P[f, \sigma_1; \varphi, \sigma_2]$ for every β belonging to $S_{\sigma_2, \varphi}$. In any case, if f and φ are represented by Taylor-D series and if we denote by $S_{f,\varphi}$ the set $S_f + S_\varphi$, then $\overline{S_{f,\varphi} \cup P[f, \sigma_1; \varphi, \sigma_2]}$ is the same as $S_{f,\varphi}$. As to general Dirichlet series, we cannot even claim that the closure of this union is equal to the closure $\bar{S}_{f,\varphi}$ of $S_{f,\varphi}$. To have composition theorems concerning general Dirichlet series without having to introduce the order of the functions (except at the singularities), the introduction of the sets $P[f, \sigma_1; \varphi, \sigma_2]$ seems natural and indispensable.

If $\sigma_1 = \sigma_2 = -\infty$, the set $P[f, \sigma_1; \varphi, \sigma_2]$ is denoted by $P[f; \varphi]$. Note that the two sets $P[f; \varphi]$ and $P[\varphi; f]$ are distinct, in general.

To grasp the meaning of a set $P[f; \varphi]$ it is helpful to say that a point γ belonging to it is in the case of a general Dirichlet series '*a fictitious sum*' *$\alpha + \beta$, where $\alpha \in S_f$, $\beta \in S_\varphi$.*

We are going to prove the following theorem.

THEOREM V.3.1. *Let $F(s)$ be a function represented by (A, Λ) $(\lambda_1 > 1)$; suppose that the function $\varphi(s)$ represented by a series (D, Λ), where $D = \{d_n\}$ $(d_n \neq 0,\ n \geqslant 1)$ is uniform and that the sequence $\{a_n d_n^{-1}\}$ admits a generating sequence $G = \{g_n\}$ of integral order k with respect to Λ. If*

$$\limsup \frac{\log |g_n|}{n} < \infty$$

and if the function $f(s)$ represented by (G, N) is uniform, then F itself is *uniform and the only possible singular points of F are the points of* $S_{f,\varphi}=\overline{S_f+S_\varphi}$, *the points of* $\overline{P[\varphi;f]}$, *and the points of* S_φ.

Suppose, first of all, that φ takes all possible values except one. By writing, as in the proof of Theorem V.2.1, $F(z)$ in a form similar to the integral in (V.3), where a_n is replaced by g_n and where b_n is replaced by d_n, we see that $\sigma_{a,F}\leqslant\sigma_{a,f}+\sigma_{a,\varphi}$ ($\sigma_{a,f}=\limsup \log|g_n|/n$). What we have to show is that $F(s)$ is holomorphic and uniform in the part of the complement of the set

$$S_{f,\varphi}\cup S_\varphi\cup\overline{P[\varphi;f]}$$

with respect to the plane that forms a domain containing a half-plane $P_{\sigma'}$. In the plane of the variable $z=x+iy$, let L be a simple Jordan arc, $L=\bar{L}\subset\Delta$, having one of its extremities z^* in the half-plane $x>\sigma_{a,f}+\sigma_{a,\varphi}$, and let $z_0\in L$.

There exist a $\delta>0$ and two distinct numbers a and b, $a=a(z_0)$, $b=b(z_0)$, such that for every $\beta\in S_\varphi$ the following relations hold: (1) $|z_0-\beta|>\delta$; for every $\rho_a\in E_\varphi(a,-\infty)$, every $\rho_b\in E_\varphi(b,-\infty)$ and every $\alpha\in S_f$

$$(2)\ |z_0-\alpha-\rho_a|>\delta,\quad (3)\ |z_0-\alpha-\rho_b|>\delta,\quad (4)\ |z_0-\beta-\alpha|>\delta.$$

First of all, it is evident that there exists a $\delta_1>0$ such that (1) and (4) are satisfied for the specified α and β with $\delta=\delta_1$; for otherwise z_0 would belong to either S_φ or to $\bar{S}_{f,\varphi}$. On the other hand, if there are no quantities a and b and a $\delta_2>0$ such that (2) and (3) hold with $\delta=\delta_2$, then for every $\delta>0$ we would have the inequality $|z-\alpha-\rho_c|\leqslant\delta$ for every c, with a certain

$$\rho_c\in E_\varphi(c,-\infty)$$

and a certain α, except possibly for at most one value $c=c(\delta)$. But clearly, $c(\delta)$ (if this value existed) would have for sufficiently small δ the same value, and z_0 would therefore belong to $P[\varphi,f]$. The smaller of the two quantities δ_1 and δ_2 would be the δ in (1), (2), (3) and (4). It follows from these four inequalities that to every $z_0\in L$ there corresponds an $\varepsilon>0$, $\varepsilon=\varepsilon(z_0)$, such that for $z\in C(z_0,\varepsilon)$

$$|z-\beta|>\varepsilon,\qquad |z-\alpha-\rho_a|>\varepsilon,$$
$$|z-\alpha-\rho_b|>\varepsilon,\qquad |z-\beta-\alpha|>\varepsilon.$$

The quantity $\varepsilon=\varepsilon(z_0)$ depends on z_0, but if we denote by $\varepsilon'(z_0)$ the greatest lower bound of all the quantities $\varepsilon(z_0)$ (for fixed z_0 in L) having the requisite properties, we see that

$$\inf_{z_0 \in L} \varepsilon'(z_0) = \eta > 0$$

(L being compact). In other words, there exists an $\eta > 0$ independent of z_0 such that for $a = a(z_0)$ and $b = b(z_0)$ every $\rho_a \in E_\varphi\,(a(z_0), -\infty)$, every $\rho_b \in E_\varphi(b(z_0), -\infty)$, every $\alpha \in S_f$ and every $\beta \in S_\varphi$ the following inequalities hold:

$$\begin{aligned} &|z - \beta| > \eta, && |z - \alpha - \rho_a| > \eta, \\ &|z - \alpha - \rho_b| > \eta, && |z - \beta - \alpha| > \eta, \end{aligned} \tag{V.14}$$

as long as $z \in \overline{C(z_0, \eta)}$ with $z_0 \in L$.

We choose finitely many points $z_0^{(1)}, z_0^{(2)}, \ldots, z_0^{(k)}$ such that $z_0^{(m+1)} \in \overline{C(z_0^{(m)}, \eta/2)}$ $(1 \leqslant m \leqslant k-1)$, $z_0^{(j)} \in L$ $(1 \leqslant j \leqslant k)$, $z_0^{(1)} = z^*$, the other extremity z_1^* of L being such that $z_1^* \in \overline{C(z_0^{(k)}, \eta/2)}$.

Now $\lim \varphi(s) = 0$ uniformly in t as $\sigma \to \infty$ (because $\lambda_1 > 0$, here we have $\lambda_1 > 1$). But it is also known that $\varphi(s) \neq 0$ for sufficiently large σ; this can be seen by writing

$$\varphi(s) = \sum d_n\, e^{-\lambda_n s} = e^{-\lambda_1 s}\Bigl(1 + \sum_{n \geqslant 2} d_n\, e^{-(\lambda_n - \lambda_1)s}\Bigr),$$

where the last sum tends to zero uniformly as $\sigma \to \infty$. Consequently, there exists a σ_0 such that

$$\Re(\rho_{a(z_0^j)}) < \sigma_0, \quad \Re(\rho_{b(z_0^j)}) < \sigma_0;$$

and if $\sigma_1 + \varepsilon$ is negative of sufficiently large absolute value and $s = \sigma_1 + \varepsilon + it$, then for every $z \in \overline{C(z_0^{(j)}, \eta)}$ $(1 \leqslant j \leqslant k)$, every $\beta \in S_\varphi$, and every $a = a(z_0^{(j)})$, $b = b(z_0^{(j)})$ $(1 \leqslant j \leqslant k)$ we have:

$$|z - s - \rho_a| > \eta, \qquad |z - s - \rho_b| > \eta, \qquad |z - s - \beta| > \eta. \tag{V.15}$$

Now let $C_1^*(\varepsilon)$ be the same contour as that defined in (V.4). It follows from (V.14) and (V.15) that if $\varepsilon > 0$ is sufficiently small, the inequalities (V.15) hold for every point s of $C_1^*(\varepsilon)$ (if σ_1 is chosen negative and of sufficiently large absolute value) and also

$$\Re(z^* - s) \geqslant \Re(z^*) - \sigma_{a,f} - 2\varepsilon > \sigma_{a,\varphi} + \varepsilon$$

(for sufficiently small ε). If s is a fixed point of $C_1^*(\varepsilon)$ and if z varies on L, the point $u = z - s$ varies on a curve $L(s)$ obtained from L by a translation. One of the extremities $u^* = z^* - s$ of $L(s)$ is in the half-plane $\sigma > \sigma_{a,\varphi} + \varepsilon$; hence we have

$$|\varphi(u^*)| = |\varphi(z^* - s)| \leqslant \sum |d_n| \, e^{-\lambda_n(\sigma_{a,\varphi} + \varepsilon)} = M$$

for every $s \in C_1^*(\varepsilon)$.

Now by Schottky's theorem we know that if $\Phi(z)$ is holomorphic on $\overline{C(\alpha, R)}$, if $|\Phi(\alpha)| < N$, if $\Phi(z)$ does not assume on this closed disk two distinct values a and b, and if $0 < \gamma < 1$, then on $\overline{C(\alpha, \gamma R)}$ we have:

$$|\Phi(z)| < K(a, b, N, \gamma) < \infty.$$

Consequently, by setting for fixed s:

$$u_2^* = z_0^{(2)} - s, \ldots, u_k^* = z_0^{(k)} - s,$$

we can write for $u \in \overline{C(u^*, \eta/2)}$

$$|\varphi(u)| < K(a(z_0^{(1)}), b(z_0^{(1)}), M, \tfrac{1}{2}) = K_1.$$

that is, also $|\varphi(u_2^*)| < K_1$. By applying Schottky's theorem again we have for $u \in C(u_2^*, \eta/2)$:

$$|\varphi(u)| < K(a(z_0^{(2)}), b(z_0^{(2)}), K_1, \tfrac{1}{2}) = K_2,$$

that is, also $|\varphi(u_3^*)| \leqslant K_2$, etc.

Hence, by applying Schottky's theorem k times we see that $|\varphi(z-s)| < B$ when z varies on a channel whose central line is $L(s)$, where s is an arbitrary point of $C_1^*(\varepsilon)$ and B is independent of z and s.

It is now easy to see that the integral (V.4) again defines a holomorphic function on L. The rest of the proof is a simple matter.

In the preceding we have assumed that $\varphi(s)$ takes all the possible values except at most one. If $\varphi(s)$ does not take two distinct values, the proof becomes easier, because if a_1 and b_1 are the two values which $\varphi(s)$ does not take, then the last passage of the proof where Schottky's theorem is applied can be used directly with $a(z_0^{(j)}) = a_1$, $b(z_0^{(j)}) = b_1$ $(1 \leqslant j \leqslant k)$.

V.4. Composition of functions of rapid growth

In this section the permitted growth of the analytic continuation of a series (A, Λ) (except at the singularities) exceeds that of every power of $|t|$. This growth can be expressed by the inequality

$$\log |f(s)| < C(|t|),$$

where $C(t)$ $(t>0)$ increases to infinity, and the only condition imposed on C (as to its growth) is the convergence of the integral of $C(t)/t^2$ over $[a, \infty[$ $(a>0)$. The convergence of such an integral plays a very characteristic role in the study of entire functions that are extensions to the complex plane of the Fourier transform of functions of compact support. The study of such entire functions is indispensable for what follows.

True, the next lemma is much more precise than the proof of the theorems on the composition of functions (and their singularities) demands; a simpler and less precise form would be sufficient by and large, but the form in which we give it here will be essential for the arguments of Chapter IX.

V.4.1. *Let $C(x)$ be a non-decreasing function that vanishes in a neighbourhood of the origin. Suppose that*

$$\int_1^\infty \frac{C(x)}{x^2}\, dx < \infty,$$

and set for $a>0$:

$$h(a) = \int_a^\infty \frac{C(x)}{x^2}\, dx.$$

Let p and q be two constants such that $p>2$, $0\leqslant q<p-2$.

For every $a>0$ there exists an even entire function $F_a(z)$ satisfying the following conditions:

$$\text{(V.16)} \quad |F_a(z)| \leqslant e^{peh(a)|y| - C(|z|)} L_a(x),$$

where $L_a(x)$ is a positive and even function decreasing for $x\geqslant 0$; $L_a \in L$, $\|L_a\|=1$; the Fourier transform $\hat{F}_a(u)$ of F_a is real, non-negative, even, and satisfies the conditions:

$$\text{(V.17)} \quad \hat{F}_a(u) = \frac{e^{-C(a-0)}}{2e\sqrt{(p+q)/(p-q-2)}} \quad \text{for} \quad |u| \leqslant qeh(a)$$

$$\text{(V.18)} \quad \hat{F}_a(u) = 0 \quad \text{for} \quad |u| \geqslant peh(a).$$

We recall that the Fourier transform of F (if $F\in L$) is given by

$$\hat{F}(u) = \int F(x)\, e^{-ixu}\, dx$$

(we do not put a factor before the integral; when the lower and the upper limits of integration are $-\infty$ and $+\infty$, respectively, we do not write them down). Finally, we recall that $\|f\|$ (for $f \in L$) denotes the norm of f in L, that is, the integral of $|f|$ over $]-\infty, \infty[$.

The proof of Lemma V.4.1 is based essentially on the following lemma.

V.4.2. *Let* $\{\mu_n\}$ *be a sequence of positive numbers such that*

$$\text{(V.19)} \qquad \sum \mu_n < \infty.$$

Then the product

$$\text{(V.20)} \qquad F(z) = \prod \frac{\sin \mu_n z}{\mu_n z}$$

converges uniformly on every compact set of the plane of $z = x + iy$; *it represents an entire even function; and if we set:*

$$\text{(V.21)} \qquad \mu = \sum \mu_n$$

$$S(u) = \inf_{n \geqslant 0} (\mu_1 \mu_2 \ldots \mu_n u^n)^{-1}$$

then[5]

$$\text{(V.22)} \qquad |F(z)| \leqslant e^{\mu|y|} S(|z|).$$

If $\hat{F}$ *is the Fourier transform of the restriction of* F *to* **R**, *then*

$$\text{(V.23)} \qquad \hat{F}(u) =$$

$$2\pi \lim_{n \to \infty} \int_{-\mu_1}^{\mu_1} dt_1 \int_{-\mu_2}^{\mu_2} dt_2 \ldots \int_{-\mu_n}^{\mu_n} f_n(u + t_1 + \cdots + t_n)\, dt_n,$$

where

$$f_n(u) = \begin{cases} \dfrac{1}{2^{n+1}\mu, \mu_2 \cdots \mu_{n+1}} & \text{for} \quad |u| < \mu_{n+1} \\ 0 & \text{for} \quad |u| \geqslant \mu_{n+1}. \end{cases}$$

With the determination $\lim_{z \to 0} \log \sin z/z = 0$ we have:

$$\log \frac{\sin z}{z} = -\frac{z^2}{6} + O(z^4) \quad (z \to 0).$$

Consequently, as z varies on a compact set D,

$$\sum \log \frac{\sin \mu_n z}{\mu_n z}$$

converges uniformly[6] by virtue of (V.19).

Since $|\sin z| \leqslant e^{|y|}$, $|\sin z| \leqslant |z|\, e^{|y|}$, we write for $1 \leqslant k \leqslant n$

$$|\sin \mu_k z| \leqslant e^{\mu_k |y|},$$

and for $k > n$ $(n \geqslant 0)$:

$$|\sin \mu_k z| \leqslant \mu_k |z|\, e^{\mu_k |y|}.$$

So we obtain for every integer $n > 0$:

$$|F(z)| = \left|\prod \frac{\sin \mu_k z}{\mu_k z}\right| \leqslant \frac{1}{\mu_1 \mu_2 \ldots \mu_n |z|^n}\, e^{\mu |y|},$$

from which (V.22) follows.

As to (V.23), we remark that when $\varphi_n(u)$ denotes the expression following 'lim' in this equality, then we have the convolution

$$\varphi_n = \varphi_{n-1} * f_n, \quad (n > 1), \quad \varphi_1 = f_1,$$

where f_n is, incidentally, the Fourier transform of $(\sin \mu_n x/\mu_n x)(2\pi)^{-1}$. Thus, $2\pi\varphi_n$ is the Fourier transform of

$$\text{(V.24)} \qquad \prod_{k \leqslant n} \frac{\sin \mu_k x}{\mu_k x}.$$

Since the product (V.24) tends to a limit $F(x)$, and since this product is bounded by a function independent of n and belonging to L, the limit in (V.23) exists and is equal to the Fourier transform of the limit of (V.24) (as $n \to \infty$), that is, to the Fourier transform of $F(x)$.

Now we proceed to the proof of Lemma V.4.1.

Let $C_a(x)$ be the function that is zero for $0 \leqslant x < a$ and equal to the restriction of $C(x)$ on $[a, \infty[$; we set $Q_a(t) = E[C_a(e^{t+1})]$ (we recall that $E[d]$ denotes the integral part of (d)). It follows from (V.15) that the integrals[7]

$$\int_{\log a - 1}^{\infty} e^{-t} Q_a(t)\, dt = \int_{\log a - 1 - 0}^{\infty} dQ_a(t)$$

converge, where the equality of the two integrals is a result, after integrating by parts, of the fact that there exists a sequence $N_j \uparrow \infty$ such that

$$Q_a(N_j)\, e^{-N_j} \to 0.$$

Now $Q_a(t)$ is a step-function whose jumps are positive integers m_k

($k \geqslant 1$); if q_k are the abscissae of these discontinuities, we set $\nu_k = e^{-q_k}$ and let $\{\mu_n\}$ be the set of the ν_k, each counted m_k times.

Then:

$$\text{(V.25)} \quad \sum \mu_n = \int_{\log a-1-0}^{\infty} e^{-t}\, dQ_a(t) = \int_{\log a-1}^{\infty} Q_a(t)\, e^{-t}\, dt$$

$$\leqslant e\int_a^{\infty} \frac{C(x)}{x^2}\, dx = eh(a).$$

We now construct the function $F(z)$ of Lemma V.4.2 with the μ_n we have just defined. We set $\beta = (p-q-2)eh(a)/2$, $\gamma = (p+q)eh(a)/2$ and

$$F_a(z) = \frac{e^{-C(a-0)}}{2\pi e} \cdot \frac{1}{\sqrt{\beta\gamma}} \cdot \frac{\sin \beta z}{z} \cdot \frac{\sin \gamma z}{z}\, F(z).$$

Then:

$$\left| \frac{\sin \gamma z}{\gamma z} \frac{\sin \beta z}{\beta z} \right| \leqslant e^{(\gamma+\beta)|y|} \min\left(1, \frac{1}{\beta\gamma |z|^2}\right)$$

$$\leqslant 2e^{(\gamma+\beta)|y|} \frac{1}{1+\beta\gamma x^2},$$

which enables us to write (for the chosen values of β and γ)

$$\text{(V.26)} \quad |F_a(z)| \leqslant \frac{1}{e}\, e^{(\beta+\gamma)|y| - C(a-0)} |F(z)| L_a(z)$$

$$= \frac{1}{e}\, e^{(p-1)eh(a)|y| - C(a-0)} |F(z)| L_a(x),$$

where $\|L_a\| = 1$.

For a given choice of the μ_n we can write (V.20) in the form

$$\log F(z) = \int_{\log a-1-0}^{\infty} \{\log(\sin e^{-t}z) + t - \log z\}\, dQ_a(t)$$

(with the evident determination of the log under the integral sign).

Since $S(|z|) \leqslant (\mu_1, \mu_2 \ldots \mu_n |z|^n)^{-1}$ no matter what n is, we can write for $y > 0$ and any $B > 0$:

$$\log|F(z)| \leqslant y \int_{\log a-1-0}^{\infty} e^{-t}\, dQ_a(t) + \int_{\log a-1-0}^{B} t\, dQ_a(t)$$

$$- \log|z| \int_{\log a-1-0}^{B} dQ_a(t).$$

By setting $B = \log |z|$ and integrating the second integral by parts, we find:

$$\begin{aligned}
\log |F(z)| &\leqslant y \int_{\log a-1-0}^{\infty} e^{-t}\, dQ_a(t) - \int_{\log a-1}^{\log |z|} Q_a(t)\, dt \\
&= y \int_{\log a-1}^{\infty} Q_a(t)\, e^{-t}\, dt - \int_{\log a-1}^{\log |z|} Q_a(t)\, dt \\
&\leqslant ey \int_{a}^{\infty} \frac{C(u)}{u^2}\, du - \int_{\log a-1}^{\log |z|} E[C(e^{t+1})]\, dt.
\end{aligned}$$

And if $|z| \geqslant a$, we can write

$$\begin{aligned}
\log |F(z)| &\leqslant ey \int_{a}^{\infty} \frac{C(u)}{u^2}\, du - \int_{\log |z|-1}^{\log |z|} E[C(e^{t+1})]\, dt \\
&\leqslant ey \int_{a}^{\infty} \frac{C(u)}{u^2}\, du - C(|z|) + 1 \\
&= eh(a)y - C(|z|) + 1.
\end{aligned}$$

This inequality combined with (V.26) yields (V.16). For $|z| < a$ (V.16) follows immediately from (V.26) and from (V.22), written down for the function F that occurs in our proof (in which we can replace $S(|z|)$ by 1; μ does not exceed $eh(a)$).

We now define Δ_γ and Δ_β as follows:

$$\Delta_\gamma(u) = 1 \quad \text{for} \quad |u| \leqslant \gamma, \qquad \Delta_\gamma(u) = 0 \quad \text{for} \quad |u| > \gamma$$

$$\Delta_\beta(u) = \frac{1}{2\beta} \quad \text{for} \quad |u| \leqslant \beta, \qquad \Delta_\beta(u) = 0 \quad \text{for} \quad |u| > \beta.$$

Then we have (with our choice of β and γ):

$$\hat{F}_a(u) = \frac{\sqrt{\beta/\gamma}}{2e}\, e^{-C(a-0)} \Delta_\gamma(u) * \Delta_\beta(u) * \hat{F}(u).$$

But it follows from (V.25) that for $\mu = \sum \mu_n$, $\gamma = (p+q)eh(a)/2$, $\beta = (p-q-2)eh(a)/2$ we have: $\gamma - \beta - \mu \geqslant q$; and by (V.25) and the definition of the f_n:

$$\Delta_\gamma(u) * \Delta_\beta(u) * \hat{F}(u) = \Delta_\gamma(u) \quad \text{for} \quad |u| \leqslant \gamma - \beta - \mu.$$

So we have, in particular, (V.17). Now (V.18) follows from (V.16) and from the Paley-Wiener theorem.[8] Thus, Lemma V.4.1 is proved.

Now we are in a position to state the following theorem.

THEOREM V.4.3. *Let* $M=\{\mu_n\}$ *be a sequence of positive step* h, *and* Λ *a sequence such that*

$$(\text{V.27}) \qquad |\lambda_n - \mu_n| \leqslant g < h/2.$$

Let $C_1(u)$ *and* $C_2(u)$ *be two increasing functions on* $[0, \infty[$ *with*

$$(\text{V.28}) \qquad \int_1^\infty \frac{C_1(u)}{u^2}\,du < \infty, \qquad \int_1^\infty \frac{C_2(u)}{u^2}\,du < \infty,$$

$C_2(u)$ *being concave.*

Suppose that $f(s)=(A, \Lambda)$ *is uniform and does not increase in* P_{σ_1}, *except at the singularities, faster than* $e^{C_1(|t|)}$, *and that* $\varphi(s)=(B, M)$ *is uniform and does not increase in* P_{σ_2}, *except at the singularities, faster than* $e^{C_2(|t|)}$.

We set $C=\{c_n\}$, *where* $c_n=a_nb_n$ $(n\geqslant 1)$ *and let* $F(s)=(C, M)$. *Then:* $\sigma_{a,F}\leqslant\sigma_{a,f}+\sigma_{a,\varphi}$. *For* $\sigma>\sigma_2+H(\sigma_1)$ *the function* $F(s)$ *is uniform, its only possible singularities in this half-plane are the points of* $\overline{S_{\sigma_1,f}+S_{\sigma_2,\varphi}}$; *and in this half-plane* $F(s)$ *does not grow, except at the singularities, faster than* $e^{C_2(|t|)}$.

The proof of this theorem does not differ materially from that of Theorem V.2.1. Taking $c>\sigma_{a,f}$ we consider this time the integral

$$(\text{V.29}) \qquad \Phi(z) = \int_{c-i\infty}^{c+i\infty} f(s)\varphi(z - s)H(-is)\,ds,$$

where $H(z)$ is an entire function of exponential type which decreases like $L(x)\,e^{-C_1(x)-C_2(x)}$, $L(x)\in L$, in every strip $|y|<d$. In the proof of Theorem V.2.1 we considered a similar integral, where instead of $H(-is)$ we had the function s^{-k-1} $(k>\nu+\mu$, ν and μ being the orders of f and of φ, respectively). It is the strong decrease of $H(-is)$ in every strip $|\sigma|<d$ (as $|t|\rightarrow\infty$) which permits us to compensate the strong increase of f and of φ in such a strip, except at the singularities; and the fact that H is an entire function makes it unnecessary to add $s=0$ to the singularities of f on integration, in other words, there is no need to introduce $S_{\sigma_2,\varphi}$ as subset of the set of possible singular points of F (as this is the case in Theorem V.2.1). Another important advantage compared with Theorem V.2.1 is the very simple expression $c_n=a_nb_n$ for the coefficients of the compositum.

On the other hand, Theorem V.2.1 has an equally important advantage over Theorem V.4.3 in the fact that no assumption is made on the sequences Λ and M; whereas in Theorem V.4.3 we assume that the step of M is positive and that (V.27) holds.

Since the change of finitely many coefficients and exponents in (A, Λ) and in (B, M) does not alter anything in the conditions nor in the conclusion of the theorem, we may assume without loss of generality that

$$\text{(V.30)} \quad \lambda_{n+1} - \lambda_n > h_1 = h - \delta > 0 \quad (n \geqslant 1),$$

where δ is sufficiently small so that also

$$\text{(V.31)} \quad |\lambda_n - \mu_n| \leqslant g < h_1/2.$$

We are now going to define the function H in (V.29). According to the conditions (V.28) and Lemma V.4.1 we may choose the constants p, q, a so that by setting

$$C(x) = C_1(x) + C_2(x),$$

there exists an even entire function $F_a(z)$ satisfying (V.16), (V.17) and (V.18) with

$$peh(a) \leqslant h_1/2$$

$$qeh(a) \geqslant g;$$

that is,

$$|F(z)| \leqslant e^{(h_1/2)|y| - C(|x|)} L(x), \quad (L(x) \in L),$$

$L(x)$ being an even function, decreasing on $[0, \infty]$,

$$\text{(V.32)} \quad \hat{F}_a(u) = P \quad \text{for} \quad |u| \leqslant g,$$

$$\text{(V.33)} \quad \hat{F}_a(u) = 0 \quad \text{for} \quad |u| \geqslant h_1/2,$$

and P a positive constant.

We now set $H(z) = F_a(z)$ and consider the integral (V.29) with $c > \sigma_{a,f}$, the integration being taken over the line $\sigma = c$. By imitating the argument in the proof of Theorem V.2.1 we see that (V.29) converges uniformly in every compact set of the half-plane $x > \sigma_{a,\varphi} + c$ and that

$$\Phi(z) = \sum b_n e^{-\mu_n z} \int_{c-i\infty}^{c+i\infty} \sum a_m e^{(\mu_n - \lambda_m)s} H(-is)\, ds.$$

We have also

$$\int_{c-i\infty}^{c+i\infty} \sum a_m e^{(\mu_n - \lambda_m)s} H(-is)\, ds$$

$$= \sum a_m \int_{c-i\infty}^{c+i\infty} e^{(\mu_n - \lambda_m)s} H(-is)\, ds,$$

and, given the properties of H, by applying Cauchy's theorem we have

$$\int_{c-i\infty}^{c+i\infty} e^{(\mu_n-\lambda_m)s} H(-is)\, ds = i\int e^{it(\mu_n-\lambda_m)} H(t)\, dt = i\hat{H}(\lambda_m-\mu_n).$$

Therefore we can write:

$$\Phi(z) = i\sum b_n\, e^{-\mu_n z} \sum a_m \hat{H}(\lambda_m - \mu_n).$$

But by (V.31), $|\mu_n-\lambda_n| < g$; hence, by (V.32):

$$\hat{H}(\mu_n - \lambda_n) = P > 0.$$

And by (V.30) and (V.31), when $m \neq n$, we have $|\lambda_m-\lambda_n| \geqslant h_1/2$, that is, $\hat{H}(\lambda_m-\lambda_n)=0$ (see (V.33)).

Thus:

$$\text{(V.34)}\qquad \Phi(z) = iP\sum a_n b_n\, e^{-\mu_n z},$$

and so

$$F(z) = -iP\Phi(z).$$

Evidently,

$$\begin{aligned}\sigma_{a,F} &= \limsup \log |c_n|/\mu_n \\ &\leqslant \limsup \log |a_n|/\lambda_n + \limsup \log |b_n|/\mu_n \\ &= \sigma_{a,f} + \sigma_{a,\varphi}.\end{aligned}$$

The rest of the proof goes as for Theorem V.2.1. We construct the same division; we define $\mathfrak{D}_1(\varepsilon)$ in the same way, with one exception: we do not attribute any special role to the point $s=0$: it only occurs if belonging to $S_{\sigma_1,f}$ like all the other points of this set.

Observe that since $C_2(u)$ is an increasing concave function for $u>0$, we have:

$$\text{(V.35)}\qquad C_2(|t-y|) \leqslant C_2(|t|+|y|) \leqslant C_2(|t|) + C_2(|y|).$$

When we now define $C_1^*(\varepsilon)$ and Δ as in the proof of Theorem V.2.1, except that in the definition of Δ the inequality

$$x > \max(H_f(\sigma_1), 0) + \sigma_2 + 2\varepsilon$$

is now replaced by

$$x > H_f(\sigma_1) + \sigma_2 + 2\varepsilon$$

and the relation

$$\gamma \in (S_{\sigma_2,f} + S_{\sigma_1,f}) \cup S_{\sigma_2,\varphi} \quad \text{by} \quad \gamma \in S_{\sigma_2,\varphi} + S_{\sigma_1,f},$$

then the arguments used in the proof of V.2.1 lead us first for $s \in C_1^*(\varepsilon)$ and $z \in \Delta$ to the inequality:

$$|\varphi(z - s)| \leqslant N\, e^{C_2(|t-y|)};$$

that is, by (V.35) to:

$$|\varphi(z - s)| \leqslant N\, e^{C_2(|t|) + C_2(|y|)};$$

and finally, for $z \in \Delta$, where Φ is holomorphic and uniform, to the inequality

$$|\Phi(z)| \leqslant P(\varepsilon)\, e^{C_2(|y|)},$$

which completes the proof of the theorem.

We conclude this section with a theorem that can be obtained from Theorem V.4.3 in exactly the same manner as Theorem V.2.8 is obtained from Theorem V.2.1.

THEOREM V.4.4. *Suppose that for Λ*

$$|\lambda_n - n| \leqslant k < \tfrac{1}{2},$$

where k is a constant, and that $f(s) = (A, \Lambda)$. Suppose also that there exists a point $\alpha = \sigma_{a,f} + it_0$ having the following properties:

(1) f is holomorphic in the set of disks $C_q = C(\alpha + 2q\pi i, \delta)$, where q is an arbitrary integer and δ a constant.

(2) There exists a positive increasing sequence $\{p_m\}$ such that in $C_{\pm m}$ $(m \geqslant 0)$: $|f(s)| \leqslant p_m$ and that

$$\sum \frac{\log p_m}{m^2} < \infty.$$

The function $F(s) = (A, N)$ (a Taylor-D series with $\sigma_{a,F} = \sigma_{a,f}$) then has an analytic continuation into the C_q. In particular, α is a regular point for F.

To obtain this theorem it is sufficient to set in V.4.3

$$\varphi(s) = \frac{1}{e^s - 1} = \sum e^{-ns},$$

where f is the function in V.4.4.

V.5. Composition of 'Hurwitz type'

First of all, we recall Hurwitz' original theorem on the composition of Taylor series.[9] We do this just as incompletely as in the statement of Hadamard's theorem in V.1.

If E_1 *is the set of singularities of the function represented by* $\sum a_n z^{-(n+1)}$, E_2 *the set of singularities of the function represented by* $\sum b_n z^{-(n+1)}$ *and* E_3 *the set of singularities of the function represented by* $\sum c_n z^{-(n+1)}$, *where* $c_n = a_n b_0 + \binom{n}{1} a_{n-1} b_1 + \cdots + a_0 b_n$, *then to every point* e_3 *of* E_3 *there corresponds a point* e_1 *of* E_1 *and a point* e_2 *of* E_2 *such that* $e_3 = e_1 + e_2$.

If $z=0$ is a regular point for the function $\psi(z)$ represented by $\sum a_n z^{-(n+1)}$, we say that the function $f(s)$ represented by $\sum a_n e^{-(n+1)s}$ is regular at $-\infty$. We say that $-\infty$ is a singularity for $f(s)$ if $z=0$ is a singularity of $\psi(z)$.

By setting $z=e^s$ Hurwitz' theorem can be stated in the following way:

If $f(s) = \sum a_n e^{-(n+1)s}$, $\varphi(s) = \sum b_n e^{-(n+1)s}$ and $F(s) = \sum c_n e^{-(n+1)s}$ with $c_n = a_n b_0 + \binom{n}{1} a_{n-1} b_1 + \cdots + a_0 b_n$, then $F(s)$ has no singularities other than the points γ of the form $\gamma = \log(e^\alpha + e^\beta)$, where α is a point of S_f and β a point of S_φ. Evidently, if $\alpha = -\infty$ is a singularity of f, this point must also be regarded as belonging to S_f, and every point $\log(e^{-\infty} + e^\beta) = \beta$ belonging to S_φ must be regarded as a possible singularity of F. Thus, if $\beta = -\infty$ is a singularity of φ, then every α of S_f is a possible singularity of F.

Let $\theta(s) = \sum d_n e^{-\lambda_n s}$ be uniform. We say that $s = -\infty$ *is a regular point of* θ if the following three conditions hold:

(1) $\theta(s)$ is holomorphic in a half-plane $\sigma < \sigma_0$,

(2) $\theta(s)$ tends to a finite limit, which we denote by $\theta(-\infty)$, uniformly in t as σ tends to $-\infty$.

(3) $[\theta(s) - \theta(-\infty)]\, e^{-s}$ tends to a finite limit, which we denote by $\theta'(-\infty)$, uniformly in t as σ tends to $-\infty$. If one of these three conditions does not hold, we say that $s = -\infty$ *is a singularity for* θ, that is: $-\infty \in S_\theta$.

We obtain a simple generalization of Hurwitz' theorem if one of the two series is a Taylor-D series, the other being of arbitrary type.

THEOREM V.5.1. *Suppose that the two series*

$$f(s) = \sum a_n e^{-(n+1)s}, \quad \varphi(s) = \sum b_n e^{-\lambda_{n+1}s}$$

are uniform in the whole plane. Let $\{l_n\}$ $(n \geqslant 1)$ *be the sequence of all the sums* $\lambda_q + m$ *(where* $q \geqslant 1$ *and* $m \geqslant 0$ *are integers), these sums being written in increasing order,*[10] *and set:*

$$d_n = \sum \sum a_m b_q A(q, m),$$

where

$$A(q, m) = \lambda_{q+1}(\lambda_{q+1} + 1) \ldots (\lambda_{q+1} + m - 1)/m! \; (q \geqslant 0, m \geqslant 1),$$
$$A(q, 0) = 1,$$

the double sum being taken over all $q \geqslant 0$, $m \geqslant 0$, *for which* $\lambda_{q+1} + m = l_{n+1}$.

We set

$$F(s) = \sum d_n e^{-l_{n+1}s}.$$

Then

$$\sigma_{\alpha,F} \leqslant \log(e^{\sigma_{a,f}} + e^{\sigma_{\alpha,\varphi}}).$$

F(s) is uniform, and the only possible singularities of F at finite distance are the points γ *of the form* $\gamma = \log(e^\alpha + e^\beta)$ *(with all determinations of* log), *where* α *and* β *are arbitrary points of* S_f *and* S_φ, *respectively, and the limit points of this set.*[11]

If $-\infty$ *is a regular point of f, then there exists a* $\delta > 0$ *such that* $|e^\alpha + e^\beta| \geqslant \delta$ *for every* $\alpha \in S_f$ *and every* $\beta \in S_\varphi$, *and if F(s) is holomorphic in a half-plane* $\sigma < \sigma_0$, *then F(s) tends to a finite limit as* σ *tends to* $-\infty$, *uniformly in t* $(s = \sigma + it)$.

Let $c > \sigma_{a,f}$ and let us show that for sufficiently large x:

$$F(z) = \frac{1}{2\pi i} \int_c^{c+2\pi i} f(s)\varphi[\log(e^z - e^s)] \, e^s \, ds, \tag{V.36}$$

where for $\sigma = c$ and for sufficiently large x

$$\begin{aligned} \varphi[\log(e^z - e^s)] &= \sum b_n (e^z - e^s)^{-\lambda_{n+1}} \\ &= \sum b_n e^{-\lambda_{n+1}z}[1 - e^{(s-z)}]^{-\lambda_{n+1}} \\ &= \sum b_n e^{-\lambda_{n+1}z} \sum A(n, m) e^{m(s-z)}, \end{aligned} \tag{V.37}$$

the integration being taken over the straight line segment joining the points c and $c+2\pi i$. If $\log|e^z-e^s| \geqslant c > \sigma_{a,\varphi}$, then the integral in (V.36) can be put in the form

$$\text{(V.38)} \quad \int_c^{c+2\pi i} f(s) \sum b_n\, e^{-\lambda_{n+1} z} \sum A(n, m)\, e^{m(s-z)}\, e^s\, ds$$

$$= \sum\sum b_n A(n, m)\, e^{-(\lambda_{n+1}+m)z} \int_c^{c+2\pi i} f(s)\, e^{(m+1)s}\, ds,$$

and since the last integral is nothing but $2\pi i a_m$, we see that (V.36) holds.

The equalities we have established remain valid when the a_n and b_n are replaced by their absolute values; consequently F converges absolutely for $x > \log(e^c + e^{\sigma_{a,\varphi}})$, and since c is arbitrary as long as $c > \sigma_{a,f}$, we see that

$$\sigma_{a,F} \leqslant \log(e^{\sigma_{a,f}} + e^{\sigma_{a,\varphi}}).$$

Let $\sigma_1 < \sigma_{a,f}$; we denote by $E(\sigma_1)$ the set of singularities of the analytic continuation of $\sum a_n z^{-n-1}$ in $|z| > e^{\sigma_1}$; for $\varepsilon > 0$, let $E(\sigma_1, \varepsilon)$ be the union of finitely many disks $C(z, \varepsilon)$ containing $E(\sigma_1)$ with $z \in E(\sigma_1)$. We write $S_{\sigma_1,f}(\varepsilon) = \log E(\sigma_1, \varepsilon)$; this is therefore the image under log of $E(\sigma_1, \varepsilon)$ ($s \in S_{\sigma_1,f}(\varepsilon)$ if $e^s \in E(\sigma_1, \varepsilon)$).

Let $D(\sigma_1, \varepsilon)$ be the largest domain composed of points of the rectangle $\sigma_1 \leqslant \sigma \leqslant c$, $0 \leqslant t \leqslant 2\pi$ that do not belong to $S_{\sigma_1,f}(\varepsilon)$ and whose boundary contains the points of the interval $\sigma = c$, $0 \leqslant t \leqslant 2\pi$ (this is possible if $\varepsilon > 0$ is sufficiently small, which we assume). We write

$$\varphi[\log(e^z - e^s)] = \psi(z, s).$$

If z is fixed with x sufficiently large, then

$$\Phi(z, s) = f(s)\psi(z, s)\, e^s$$

is a holomorphic function of s on $\overline{D(\sigma_1, \varepsilon)}$, where $\psi(z, s)$ is the analytic continuation along curves (on which only s varies) situated in $\overline{D(\sigma_1, \varepsilon)}$, of the function given on $\sigma = c$ by (V.37). The function Φ (for fixed z) takes the same value at the points σ' and $\sigma' + 2\pi i$ belonging to the boundary $B(\sigma_1, \varepsilon)$ of $D(\sigma_1, \varepsilon)$.

If we denote by $B^* = B^*(\sigma_1, \varepsilon)$ the boundary of $B(\sigma_1, \varepsilon)$ composed of the points that do not belong to $\sigma = c$ nor to the lines $t=0$ and $t=2\pi$ (on which Φ, for given z, takes the same value by virtue of the manner in which $S_{\sigma_1,f}(\varepsilon)$ and $D(\sigma_1, \varepsilon)$ have been constructed), then by Cauchy's theorem we have

$$\text{(V.39)} \qquad F(z) = \frac{1}{2\pi i} \int_{B^*} f(s)\psi(z, s)\, e^s\, ds.$$

Let Δ be a compact domain containing points of the half-plane $x > \log(e^c + a^{\sigma_{a,\varphi}})$, such that for every $z \in \Delta$, every $s \in B^*$, and every $\beta \in S_\varphi$, except for $\beta = -\infty$ in case $-\infty \in S$, we have:

$$\text{(V.40)} \qquad |z - \log(e^s + e^\beta)| > \varepsilon_1$$

$$\text{(V.41)} \qquad |z - s + 2k\pi i| > \varepsilon_2 \quad (k \text{ any integer}),$$

where ε_1 and ε_2 are two given positive quantities, and all the determinations of the logarithms are taken.

Hence there exists a $\delta_1 > 0$, an $\varepsilon > 0$, and an $N > -\infty$ $(\delta_1 = \delta_1(\varepsilon_1)$, $\varepsilon = \varepsilon(\varepsilon_1)$, $N = N(\varepsilon_2))$, such that for $z \in \Delta$, $s \in B^*$, $\beta \in S_\varphi$ $(\beta \neq -\infty)$ we have:

$$|e^z - e^s - e^\beta| > \delta_1$$

$$|\log(e^z - e^s) - \beta| > \varepsilon$$

$$\mathscr{R} \log(e^z - e^s) = \log|e^z - e^s| > N.$$

The integrand in (V.39) is a holomorphic function of z and of s provided that $z \in \Delta$, $s \in B^*$; therefore F is a holomorphic function in Δ. Since ε_1 and ε_2 can be chosen arbitrarily small and σ_1 negative, of arbitrarily large absolute value, we see that F is holomorphic in every compact domain containing points of a half-plane $\sigma > \log(e^c + e^{\sigma_{a,\varphi}})$, but not containing any point of the form $\log(e^\alpha + e^\beta)$. Hence if $s = -\infty$ is a singularity for φ, the theorem is proved.

Suppose now that $s = -\infty$ is a regular point for φ. If we define Δ as above, but omitting the condition (V.41), we can show that F is holomorphic in Δ.

In fact, $\psi(z, s)$ is holomorphic as a function of z in Δ if for $z \in \Delta$, $z = s$, we give to ψ the value of $\varphi(-\infty)$; for ψ is then a continuous function of z for fixed s, and this function is differentiable with respect to z as long as $z \in \Delta$. This is evident for $z = z_0 \in \Delta$, with $z_0 \neq s \pmod{2\pi i}$.

And if $z_0=s \pmod{2\pi i}$, then:

$$\text{(V.42)}\quad \left.\frac{\partial\psi(z,s)}{\partial z}\right|_{z=z_0} = \lim_{z\to z_0}\frac{\psi(z,s)-\varphi(-\infty)}{z-z_0}$$

$$= \lim_{z\to z_0}\frac{\psi(z,s)-\varphi(-\infty)}{e^z-e^{z_0}}\cdot\lim_{z\to z_0}\frac{e^z-e^{z_0}}{z-z_0} = \varphi'(-\infty)\,e^{z_0}.$$

From the fact that ψ is holomorphic in Δ for every $s\in B^*$ it therefore follows that F is holomorphic in Δ.

Suppose now that $-\infty$ is a regular point of f.

Observe that the set $(S_{\sigma_1,f})$, which is the subset of $S_{\sigma_1,f}$ contained in the strip $0\leqslant t\leqslant 2\pi$, contains the points of the interval $L(\sigma_1)$ given by $\sigma=\sigma_1$, $0\leqslant t\leqslant 2\pi$. Let $\sum(\sigma_1,\varepsilon)$ be the union of the images under log of the disks $C(z,s)$ forming part of $E(\sigma_1,\varepsilon)$ and containing the points of $L(\sigma_1)$. Since $-\infty$ is a regular point of f, we see that it suffices for $-\sigma_1$ to be sufficiently large ($\varepsilon>0$ fixed) for the following facts to hold: on the one hand, $\sum(\sigma_1,\varepsilon)$ is separated from the images (under log) of the other disks of $E(\sigma_1,\varepsilon)$, and on the other hand, f is holomorphic on $\sum(\sigma_1,\varepsilon)$. Given that $\varepsilon>0$ is chosen sufficiently small and that such a choice of σ_1 has been made, let $l=l(\sigma_1,\varepsilon)$ be the boundary of $\sum(\sigma_1,\varepsilon)$ and $C^*=C^*(\sigma_1,\varepsilon)=$ $=B^*(\sigma_1,\varepsilon)-l(\sigma_1,\varepsilon)$. By (V.39) we can then write:

$$\text{(V.43)}\quad F(z)=\frac{1}{2\pi i}\int_l f(s)\psi(z,s)\,e^s\,ds+\frac{1}{2\pi i}\int_{C^*} f(s)\psi(z,s)\,e^s\,ds.$$

We see that the last conditions of the theorem are satisfied and that the first integral in (V.43) vanishes for sufficiently large x, because then the function to be integrated over l is holomorphic as a function of s on $\overline{\sum(\sigma_1,\varepsilon)}$ of which l is the boundary. Consequently we have

$$\text{(V.44)}\quad F(z)=\frac{1}{2\pi i}\int_{C^*} f(s)\psi(z,s)\,e^s\,ds.$$

Now this expression tends to

$$\frac{1}{2\pi i}\int_{C^*} f(s)\varphi(\pi i+s)\,e^s\,ds$$

uniformly in y, as x tends to $-\infty$.

This completes the proof of the theorem.

It is easy to verify that Theorem V.5.1 contains as a particular case Hurwitz' theorem in its most complete form (at least for uniform functions): $\lambda_n = n$ $(n \geqslant 1)$. It is enough to note that the assertion that the analytical continuation of $\mathscr{F}(z) = \sum d_n z^{-n-1}$ is holomorphic for $0 < |z| < \eta$ and tends to a finite limit as z tends to zero is equivalent to the fact that this function is holomorphic in $|z| < \eta$.

A theorem of a more general character concerning two Dirichlet series (A, Λ) and (B, M) of general type, the exponents of the series being distinct *a priori*, can be obtained by introducing the notion of the order of each function (except at the singularities).

THEOREM V.5.2. *Let $f(s) = (A, \Lambda)$ be uniform of order ν in P_{σ_1}, except at the singularities; and let $\varphi(s) = (B, M)$ be uniform of order μ in P_{σ_2}, except at the singularities. Let k be an integer greater than $\nu + \mu$, and let $a_T^{(k)}(n)$ be the n-th Taylor coefficient of f of order k. Let $L = \{l_n\}$ $(n \geqslant 1)$ be the sequence of all the sums $\lambda_q + m$ $(q \geqslant 1, m \geqslant 1)$, where the m are positive integers and the sums are written in increasing order,*[12] *and set*

$$d_n = \sum \sum a_T^{(k)}(m) b_q A(q-1, m),$$

the $A(q, m)$ being defined as in V.5.1 and the double sum taken over all $q \geqslant 2$, $m \geqslant 1$, for which $\lambda_{q-1} + m = l_n$.

We write

$$\max(\sigma_{a,f}, 0) = \sigma_f$$

$$\max(H_f(\sigma_1), 0) = \sigma'_f.$$

Let[13] *$F(s) = (D, L)$ $(D = \{d_n\})$. Then*

$$\text{(V.45)} \qquad \sigma_{a,f} \leqslant \log(e^{\sigma_f} + e^{\sigma_{a,\varphi}}),$$

F is uniform in the half-plane $\sigma > \log(e^{\sigma'_f} + e^{\sigma_2})$, and in this half-plane the only possible singularities of F are the points of the form

$$\gamma = \log(e^{\alpha} + e^{\beta}), \qquad \gamma = \log(e^{\beta} + 1)$$

where $\alpha \in S_{\sigma_1, f}$ and $\beta \in S_{\sigma_2, \varphi}$, and the limit points of this set of points (with all determinations of log).

Let $c > \max(0, \sigma_{a,f})$ and let us show that

$$\text{(V.46)} \qquad F(z) = \frac{k!}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s) \varphi[\log(e^z - e^s)] \frac{ds}{s^{k+1}},$$

where $\varphi[\log(e^z - e^s)] = \psi(z, s)$ is given by (V.37) for sufficiently large x.

In fact, the expression on the right-hand side of (V.46) can be written (see (I.3.2)) for sufficiently large x as:[14]

$$\begin{aligned}\frac{k!}{2\pi i}\int_{c-i\infty}^{c+i\infty} & f(s)\sum b_n\, e^{-\lambda_n z}\sum A(n-1,m)\, e^{m(s-z)}\,\frac{ds}{s^{k+1}}\\ &= \sum b_n \sum A(n-1,m)\, e^{-(\lambda_n+m)z}\,\frac{k!}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\, e^{ms}\,\frac{ds}{s^{k+1}}\\ &= \sum\sum a_T^{(k)}(m) b_n A(n-1,m)\, e^{-(\lambda_n+m)z} = \sum d_n\, e^{-l_n z}.\end{aligned}$$

These inequalities are valid for $x > \log(e^c + e^{\sigma_{a,\varphi}})$. They are valid for the same values of x when the a_n and b_n are replaced, respectively, by $|a_n|$ and $|b_n|$. This proves that (D, L) converges absolutely for the same values of z. But since c is chosen only to satisfy the inequality $c > \max(0, \sigma_{a,f})$, we see that (V.45) holds.

Now let $c > \max(0, \sigma_1, \sigma_{a,f})$. If we define $C_1^*(\varepsilon)$ just as we have done it in writing (V.4), we see, as in the proof of (V.4), that

$$\text{(V.47)}\qquad F(s) = \frac{k!}{2\pi i}\int_{C_1^*(\varepsilon)} f(s)\varphi[\log(e^z - e^s)]\,\frac{ds}{s^{k+1}}.$$

Now let Δ be a compact domain containing points with $x > \log(e^c + e^{\sigma_{a,\varphi}})$ and such that for an arbitrary $\varepsilon > 0$, for every $z \in \Delta$, every $\alpha \in S_f$ and every $\beta \in S_\varphi$ we have:

$$|z - \log(e^\alpha + e^\beta)| > \varepsilon$$

(no matter what determination of the logarithm is taken).

Suppose also that for $z \in \Delta$

$$\text{(V.48)}\qquad x > \log(e^{\sigma_f' + 2\varepsilon} + e^{\sigma_2}).$$

We see that to every $s \in C_1^*(\varepsilon)$ there corresponds an $\alpha \in S_{\sigma_1, f}$ such that

$$|e^s - e^\alpha| < \eta = \eta(\varepsilon),$$

where $\eta(\varepsilon)$ tends to zero with ε.

From the various inequalities above it follows, on the one hand, that there exists a $\delta > 0$ such that for sufficiently small ε the following inequality holds for every $z \in \Delta$, $s \in C_1^*(\varepsilon)$ and every $\beta \in S_{\sigma_2, \varphi}$:

$$|\log(e^z - e^s) - \beta| > \delta.$$

On the other hand, it follows from (V.48) that for $z \in \varDelta$, $s \in C_1^*(\varepsilon)$:

$$\text{(V.49)} \quad \mathscr{R}[\log(e^z - e^s)] = \log|e^z - e^s| \geqslant \log(e^x - e^\sigma) \geqslant \log(e^x - e^{\sigma_f' + 2\varepsilon}) > \sigma_2.$$

In other words, if $z \in \varDelta$, $s \in C_1^*(\varepsilon)$ and if $\delta > 0$ is given, then for sufficiently large $|\arg(e^z - e^s)|$

$$|\varphi[\log(e^z - e^s)]| < N(\delta)|\arg(e^z - e^s)|^{\mu+\delta},$$

where $\arg(e^z - e^s)$ is well-defined, because we see according to the expansion we have given for $(e^z - e^s)^{-\lambda_n}$ that $\log(e^z - e^s)$ is taken real for real z and s, and that for the other pairs of z and s this function is defined by analytic continuation.

But from the fact that $\varDelta$ is a compact set it follows that there exists a constant $A < \infty$ such that for $z \in \varDelta$, $s \in C_1^*(\varepsilon)$ and sufficiently large $|t|$:

$$|\arg(e^z - e^s)| < A|\arg e^s| = A|t|.$$

Hence, on the one hand, to every $\delta_2 > 0$ there corresponds an $N_2 = N_2(\delta_2)$ such that for sufficiently large $|t|$

$$|\varphi[\log(e^z - e^s)]| \leqslant N_2(\delta_2)|t|^{\mu+\delta_2},$$

and on the other hand, to every $\delta_1 > 0$ there corresponds an $N_1(\delta_1)$ such that for $s \in C_1^*$ and sufficiently large $|t|$

$$|f(s)| \leqslant N_1(\delta_1)|t|^{\nu+\delta_1}.$$

Therefore, if δ_1 and δ_2 are chosen sufficiently small, then (V.47) converges uniformly in $\varDelta$ and defines there a holomorphic function.

Our theorem is now proved.

Theorem V.5.2 can be strengthened in the following manner:

THEOREM V.5.3. *If $s = 0$ is a regular point of $f(s)$, then under the same assumptions and the same notation as in Theorem* V.5.2 *the following conclusion holds:*

$$F(s) = \sum d_n e^{-l_n s}$$

is uniform in the half-plane P:

$$\sigma > \log(e^{\sigma_f''} + e^{\sigma_2}), \quad \text{where} \quad \sigma_f'' = H_f(\sigma_1),$$

and the only possible singularities of $F(s)-I(s)$, *where*

$$\begin{aligned} I(s) = f(0) &\left.\frac{d^k\varphi[\log(e^s-e^\xi)]}{d\xi^k}\right|_{\xi=0} \\ &+ \binom{k}{1}f'(0)\left.\frac{d^{k-1}\varphi[\log(e^s-e^\xi)]}{d\xi^{k-1}}\right|_{\xi=0} \\ &+ \cdots + f^{(k)}(0)\varphi[\log(e^s-1)], \end{aligned}$$

in the half-plane P *are the points* γ *of the form* $\gamma=\log(e^\alpha+e^\beta)$, *where* α *and* β *are defined as in Theorem* V.5.2, *and the limit points of this set* (*all the determinations of the logarithm must be taken*).

The passage from Theorem V.5.2 to Theorem V.5.3 is analogous to that from Theorem V.2.1 to Theorem V.2.2.

Remark. If the functions f and φ occurring in the various 'composition theorems' of this chapter are not assumed to be uniform, then the corresponding statements still remain valid, provided that we denote by $S_{\sigma_1,f}$ and $S_{\sigma_2,\varphi}$ the singular sets of f and φ (in the half-planes P_{σ_1} and P_{σ_2}, respectively) to which we add cuts which make these functions uniform in the exterior of the sets so obtained. But in the conclusions we can no longer claim that the composite function F is itself uniform in the corresponding domain.

NOTES

[1] A great many authors have assumed erroneously that the statement which follows is valid in all cases.

[2] $\Delta(\sigma_1, \varepsilon)$ is clearly closed and connected and contains a half-plane $\sigma>\sigma_2$.

[3] If A and B are two point sets, then $A+B$ denotes the set of all the points $\alpha+\beta$ such that $a\in A$, $\beta\in B$.

[4] If $\lambda_n=p_n$ the expression $\sin \pi\lambda_n/\lambda\pi_n[1-\lambda_n^2/p_n^2]$ must be taken as representing

$$\prod_{m\neq p_n}(1-p_n^2/m^2).$$

[5] For $n=0$, by definition $(\mu_1\ldots\mu_n u^n)^{-1}$ is equal to 1. Thus, $S(u)\leqslant 1$.

[6] On D this sum only begins with the term n_0 $(n\geqslant n_0)$ so that $\sin \mu_n z$ does not vanish. The determination of log is evident.

[7] For a Stieltjes integral $\int_{a-0}^{b}$ denotes the limit of $\int_{a-\varepsilon}^{b}$ if ε tends to zero through positive values.

[8] We recall the Paley-Wiener theorem (at least that part which we need): if $F(x)$ belongs to L on $\mathbf{R}$, then a necessary and sufficient condition for F to be the restriction of an entire function $F(z)$ satisfying the inequality $|F(z)|\leqslant A\,e^{a|y|}$ (where A is a constant) is that the support of $\hat{F}(u)$ should belong to the interval $[-a, a]$.

[9] In the following statement, as in Theorem V.5.1 the summations $\sum$ extend over $n \geqslant 0$ (and not over $n \geqslant 1$, which is our convention if no limit is stated).

[10] In this theorem, as in Theorem V.5.2, the l_n are arranged in non-decreasing order rather than in increasing order, because two pairs (q_1, m_1) and (q_2, m_2) may yield the same $l_n = \lambda_{q1} + m_1 = \lambda_{q2} + m_2$. In the series for F the same exponent can therefore occur several times. But this does not affect the proof at all.

[11] If $-\infty \in S_f$, then every $\gamma = \log(e^{-\infty} + e^{\beta}) = \beta + 2k\pi i$ must be regarded as a possible singularity of F. Also, if $-\infty \in S_\varphi$, then every $\alpha + 2k\pi i$, where $\alpha \in S_f$, must be regarded as a possible singularity of F.

[12] See Note 10.

[13] See Note 9.

[14] We recall that $a_T^{(k)}(m) = 0$ for $m \leqslant \mu_1$.

CHAPTER VI

SOME APPLICATIONS OF THE PRINCIPLES FOR ANALYTIC CONTINUATION

VI.1. Arithmetic properties of the exponents and the analytic continuation

In this section we give an account of some applications of the arguments in II.1. Our aim is to show that the exponents of a series $f(s)=(A, \Lambda)$ which are 'distinguished' from the other exponents, for example, by their fractional parts, play a role which one could describe as 'autonomous' in the determination of the properties of the analytic continuation of the series. The coefficients of the series corresponding to these exponents already indicate certain important characteristic features of the continuation, which remain in force, no matter what the values of the other coefficients are. Thus, if $\Lambda^* \subset \Lambda$ has the property that every $\lambda_{n_j} \in \Lambda^*$ has a fractional part different from that of all the other $\lambda_n \in \Lambda$, $n \neq n_j$, then the order of magnitude of the a_{n_j} determines a set of domains (obtained from any one by all the translations $2k\pi i$, k being an arbitrary integer), where f cannot be holomorphic and bounded.

One could compare the following theorems with those of Chapter VII, which while providing conclusions concerning the analytic continuation starting out from the arithmetical character of a subset of exponents, are nevertheless of an entirely different nature.

THEOREM VI.1.1. *Suppose that $f(s)=(A, \Lambda)$ with $\sigma_{a,f}=0$ and that $\Lambda^* \subset \Lambda$ is such that every $\lambda_{n_j} \in \Lambda^*$ has one of the following properties:*

(1) $\lambda_{n_j} \neq \lambda_n \pmod 1$ *for every $n \neq n_j$;*

(2) *λ_{n_j} is an integer, and if there is another integer $\lambda_n \in \Lambda$, then one of the two inequalities $\lambda_{n_j} > 2\lambda_n$ or $\lambda_n > 2\lambda_{n_j}$ holds.*

We set[1]

$$\text{(VI.1)} \qquad \limsup \frac{\log |a_{n_j}|}{\lambda_{n_j}} = \sigma_1,$$

and assume that $\sigma_1 > -\log 2$. Let t_0 be arbitrary real, and

$$0 < \varepsilon < \log 2 + \sigma_1;$$

we denote by $\mathfrak{D}$ *the rectangle*

$$\sigma_1 - \varepsilon < \sigma < \varepsilon, \; |t - t_0| < 3 \operatorname{arc\,cos} e^{\sigma_1 - \varepsilon} \leqslant \pi.$$

The function $f(s)$ *cannot be holomorphic and bounded in the union of* $\mathfrak{D}$ *with all the rectangles obtained by the translations* $\mathfrak{D} + 2k\pi i$ *(where* k *is an arbitrary integer).*

We set

$$F(s) = \sum a_n \, e^{-\lambda_n(\sigma_1 - \varepsilon + it_0)} \, e^{-\lambda_n s} = \sum c_n \, e^{-\lambda_n s} = (C, \Lambda),$$

and denote by D the domain in the plane $z = r\,e^{i\theta}$ which consists of the union of the disk $|z| < e^{\sigma_1}$ with the domain defined by $r < 1$, $|\theta| < 3 \operatorname{arc\,cos} e^{\sigma_1 - \varepsilon}$.

One easily verifies that the starred curve Γ^* defined by $r = \cos \theta/3$ ($|\theta| < \pi$) (see II.1.3) is in the interior of D apart from the point $z = 1$, and in particular, that the part of Γ^* for which $|\theta| \geqslant 3 \operatorname{arc\,cos} e^{\sigma_1 - \varepsilon}$ is situated in the closed disk $|z| \leqslant e^{\sigma_1 - \varepsilon}$.

If the conclusion of the theorem were not satisfied, that is, if $f(s)$ were holomorphic and bounded in the union of the rectangles defined in the statement of the theorem, then there would exist a starred curve G^* sheltering Γ^* such that (C, Λ) would be holomorphic and bounded in $\mathscr{L}D(G^*)$.

Suppose, for example, that the condition (1) is satisfied for all λ_{n_j} belonging to Λ^*. By applying Theorem II.1.3 we would obtain

$$|c_{n_j}| = |a_{n_j}| \, e^{-\lambda_{n_j}(\sigma_1 - \varepsilon)} \leqslant B_1 \sup_{s \in \mathscr{L}D(G^*)} |F(s)| = P,$$

where P is a constant, and this would give

$$\limsup \frac{\log |a_{n_j}|}{\lambda_{n_j}} \leqslant \sigma_1 - \varepsilon;$$

this is a contradiction to (VI.1). The arguments used in the proofs of Theorems II.1.3 and II.1.4 allow us to draw the same conclusion as long as we assume that one of the conditions (1) or (2) is satisfied for every $\lambda_{n_j} \in \Lambda^*$.

Theorem VI.1.1 immediately suggests the following more general theorem:

THEOREM VI.1.2. *Let* $f(s) = (A, \Lambda)$ *with* $\sigma_{a,f} = 0$. *Let* Γ^* *be a starred curve defined by* $r = r(\theta)(|\theta| \leqslant \pi)$, *where* $r(\theta)$ *is a (strictly) decreasing curve on*

$[0, \pi]$, with $r(\pi) = a$ $(0 < a < 1)$, and let

$$\theta(r) \qquad (r \in [a, 1])$$

be the inverse function to $r(\theta)$. Let $\Phi(z)$ be the function attached to Γ^*, and let $\Lambda^* \subset \Lambda$ be the set of λ_n each of which is distinct modulo 1 *from all the other members of Λ with respect to Φ.*

We set (VI.1) *and assume that $\sigma_1 > \log a$. Let t_0 be arbitrary real and $0 < \varepsilon < \sigma_1 - \log a$.*

The function $f(s)$ cannot be holomorphic and bounded in the union of the rectangle $\mathfrak{D}$ defined by

$$\sigma_1 - \varepsilon < \sigma < \varepsilon, \qquad |t - t_0| \leqslant \theta(e^{\sigma_1 - \varepsilon})$$

with all the rectangles obtained by the translations $\mathfrak{D} + 2k\pi i$ (where k is an arbitrary integer).

To prove this theorem it is sufficient to copy the proof of Theorem VI.1.2, making use of the new curve Γ^*.

From these two theorems we obtain, in particular, the following theorem:

THEOREM VI.1.3. *If we add to the conditions of one of Theorems* VI.1.1 *or* VI.1.2 *the condition $\sigma_1 = 0$, then for any real t_0 and $\varepsilon > 0$ the function $f(s)$ cannot be holomorphic and bounded in the set of disks*

$$C[(t_0 + 2k\pi)i, \varepsilon]$$

(where k is an arbitrary integer).

So we find, for example, the very special case of the classical theorem of Hadamard: if the λ_n are integers and $\lambda_{n+1}/\lambda_n > 2$, then the Taylor series $\sum a_n z^{\lambda_n}$ of finite radius of convergence has its circle of convergence as natural boundary. But evidently we can extend in II.1 the very definition of the notion 'λ and λ' are distinct modulo 1 with respect to Φ', replacing it by 'λ and λ' are distinct modulo $1/p$ (where p, for example, is a positive integer) with respect to Φ'. This will be the case if for any m and m' such that

$$d_m^{(\lambda)} \neq 0, \qquad d_{m'}^{(\lambda')} \neq 0$$

(see II.1) we always have: $p(\lambda - \lambda') \neq m - m'$. If we then replace in all the arguments of II.1 $\Phi_1(z) = z\Phi(z)$ by $\Phi_p(z) = z^p\Phi(z)$, we see easily that the theorems in this part of Chapter II can be generalized. (To simplify the exposition we have only treated the particular case $p = 1$.) The theorems we have just proved here can therefore also be generalized in a corresponding manner. The very special case of the theorem we have just quoted then becomes Hadamard's theorem, as it was originally stated: if

$$\lambda_{n+1}/\lambda_n \geqslant \lambda > 1 \quad (\lambda_n \text{ are integers}),$$

then the circle of convergence is a natural boundary for the Taylor series $\sum a_n z^{\lambda_n}$. And this theorem only appears as a very special case a fairly general phenomenon.

We do not dwell here on the fact, because in the next section we rediscover the theorem of Fabry-Pólya, which has a bearing on general Dirichlet series, as a very special case of theorems obtained by starting out from the arguments of II.2.

VI.2. Analytic continuation of general Dirichlet series

The first result of this section concerns the series (A, Λ) whose abscissa of convergence is finite, Λ being of finite step: by the theorems of II.2 these series therefore have singularities. Then we consider series whose abscissa of convergence is $-\infty$ and which consequently represent entire functions.

The following theorem is an immediate consequence of Theorem II.2.3.

Theorem VI.2.1. *Suppose that Λ is of positive step h. Let $D^{\bullet}$ be its upper density. Let $f(s) = (A, \Lambda)$ with $-\infty < \sigma_{a,f} < \infty$.*

For $\alpha > 0$, $\beta \geqslant 0$ there exists a continuous function $A(\alpha, \beta)$ with $A(\alpha, 0) = 0$ such that $f(s)$ has for every real value of t_0 a singular point in the rectangle

$$\sigma_{a,f} - A(h, D^{\bullet}) \leqslant \sigma \leqslant \sigma_{a,f}, \ |t - t_0| \leqslant \pi D^{\bullet}.$$

For $A(\alpha, \beta)$ we can take the following function:

$$A(\alpha, \beta) = \pi\beta - [3 \log(\alpha\beta) - 8.5]\beta \quad \text{for} \quad \beta > 0$$
$$A(\alpha, 0) = 0.$$

If $D^{\bullet} = 0$, the expression 'singular point in the rectangle . . .' indicates that $\sigma_{a,f} + it_0$ is itself a singular point. This means that we have the following theorem:

Theorem VI.2.2. *If $h > 0$, $D^{\bullet} = 0$, $-\infty < \sigma_{a,f} < \infty$, then every point on the axis of convergence is a singular point of f.*

One expresses this fact by saying that the axis of convergence of (A, Λ) is a natural boundary for (A, Λ).

Theorem VI.2.2, which is due to Pólya, generalizes to Dirichlet series the classical theorem of Fabry concerning Taylor series: *the circle of convergence of a Taylor series* $\sum a_n z^{\lambda_n}$ *(where* λ_n *are integers) with* $\lim n/\lambda_n = 0$ *is a natural boundary*, a theorem which in turn generalizes Hadamard's theorem of which we have talked in the preceding section.

Evidently we can choose for $A(\alpha, \beta)$ $(\beta > 0)$ every function of the form

$$A(\alpha, \beta) = \pi\beta + [A(a) - 2a \log(\alpha\beta)]\beta = \pi\beta + L(a, \alpha, \beta),$$

where a is an arbitrary real number greater than 1 and $A(a)$ is the function

$$A(a) = a\{4 + a + \log[(a^2 - 1)/a^3] + \log[(a + 1)/(a - 1)]/a\}$$

(see II.2.3).

We have seen that the expression for $L(a, h, D^{\cdot})$ and consequently also that for $A(h, D^{\cdot})$ contains the term $\log(hD^{\cdot})$. Since $hD^{\cdot} \leqslant 1$, the expression $-2aD^{\cdot} \log(hD^{\cdot})$ is positive, and for fixed h (for example $h = 1$) tends to zero (as $D^{\cdot}$ tends to zero), like $-2aD^{\cdot} \log D^{\cdot}$. Therefore the expressions $L(a, h, D^{\cdot})$ and $A(h, D^{\cdot})$ tend to zero in the same manner as $D^{\cdot}$ tends to zero (for fixed h). We wish to show that the speed with which these expressions tend to zero is very characteristic. In fact, for fixed h we necessarily have

$$\liminf_{D^{\cdot} \to 0} A(h, D^{\cdot})/(-D^{\cdot} \log D^{\cdot}) = \liminf_{D^{\cdot} \to 0} L(a, h, D^{\cdot})/(-D^{\cdot} \log D^{\cdot}) \geqslant 1.$$

In other words, in $A(\alpha, \beta)$, as it occurs in Theorem VI.2.1, the number 3 cannot be replaced by a number smaller than 1. *A fortiori* it is impossible to take for $A(h, D^{\cdot})$ (with h fixed) an expression of the form $cD^{\cdot}$, where c is independent of $D^{\cdot}$.

In fact, we can prove the following theorem:

Theorem VI.2.3. *Let* $p \geqslant 2$ *be an integer and let* $\{\mu_n\}$ *be a sequence of positive integers for which*

$$\mu_{n+1}/\mu_n > p/(p - 1), \qquad \lim \Big(\sum_{k \leqslant n} \mu_k\Big)\Big/\mu_{n+1} = 0.$$

Let $\{b_n\}$ *be a sequence for which*

$$\text{(VI.2)} \qquad \limsup |b_n|^{1/n} = 1.$$

The series

$$\sum b_n \left(\frac{e^{-ps} - e^{-(p-1)s}}{2}\right)^{\mu_n}$$

converges for $\sigma > 0$, *it represents a holomorphic function* $f(s)$, *which is also represented in the same half-plane by a series* (A, Λ) *having the following properties:*

(1) *the* λ_n *are integers,* $h=1$,
(2) $\sigma_{a,f}=0$,
(3) $D^{\cdot}=1/p$,
(4) $f(s)$ *is holomorphic in the half-strip* $|t| \leqslant \pi D^{\cdot}$, $\sigma > -\mu D^{\cdot}$, *where* μ *is the positive root of*

$$\text{(VI.3)} \qquad D^{\cdot 2} e^{2\mu}(\mu^2 + \pi^2) = 4.$$

As p tends to infinity, the quantity μ *so defined satisfies the relation*

$$\text{(VI.4)} \quad \mu = -\log D^{\cdot} - \log(-\log D^{\cdot}) + \log 2 + o(1) \; (D^{\cdot} = 1/p \to 0).$$

Remark. In comparing this theorem with Theorem II.2.3 we see that for every choice of $A(\alpha, \beta)$ corresponding to the statement of Theorem VI.2.1, as $D^{\cdot}$ tends to zero, we have:

$$A(1, D^{\cdot}) \geqslant -[\log D^{\cdot} + \log(-\log D^{\cdot}) - \log 2 + o(1)]D^{\cdot} \sim -D^{\cdot} \log D^{\cdot}.$$

Also, in the last paragraph of II.2.3 the phrase 'for which the abscissa of the centre of a disk belonging to it is smaller than $[\ldots]D^{\cdot}+\sigma_c$' becomes incorrect when the number 3 in $[\ldots]$ is replaced by a number smaller than 1.

Now we proceed to the proof of the theorem.

(1) and (2) are evident: it is sufficient to argue about the series

$$\sum b_n[(z^p - z^{p-1})/2]^{\mu_n},$$

and to take (VI.2) into account.

To see (3) we remark that when we denote by $\mathfrak{N}(x)$ $(x>0)$ the number of λ_n not exceeding x, we have[2]

$$\mathfrak{N}(p\mu_n) = \sum_{k \leqslant n} (\mu_k + 1) = n + \sum_{k \leqslant n} \mu_k,$$

hence

$$\frac{\mathfrak{N}(p\mu_n)}{p\mu_n} = \frac{n}{p\mu_n} + \frac{1}{p} + \sum_{k \leqslant n-1} \mu_k (p\mu_n)^{-1}.$$

That is (evidently $\lim n/\mu_n = 0$):

$$D^{\cdot} \geqslant \limsup \mathfrak{N}(p\mu_n)/(p\mu_n) = \frac{1}{p}.$$

We also have for $p\mu_{n-1} < \lambda_k \leqslant p\mu_n$ $(n \geqslant 2)$

$$\frac{k}{\lambda_k} \leqslant \frac{\mathfrak{N}(p\mu_n)}{p\mu_n},$$

and consequently:

$$D^{\cdot} = \limsup \frac{k}{\lambda_k} = \frac{1}{p}.$$

Now we prove (4). First of all we note that for $0 < D < 1$ and $\mu \geqslant 0$: $\cos \pi D > 1 - (\pi D^2)/2$, $1 - e^{-\mu D} < \mu D$; we can therefore write

$$\text{(VI.5)} \quad \begin{aligned} &e^{2\mu}(1 + e^{-2\mu D} - 2\,e^{-\mu D} \cos \pi D) \\ &\quad < e^{2\mu}[1 + e^{-2\mu D} - 2\,e^{-\mu D} + e^{-\mu D}(\pi D)^2] \\ &\quad < e^{2\mu}[(1 - e^{-\mu D})^2 + (\pi D)^2] < e^{2\mu}(\mu^2 + \pi^2) D^2. \end{aligned}$$

If $s \in S \equiv \{s \,|\, |t| < \pi D^{\cdot},\ -\mu D^{\cdot} \leqslant \sigma \leqslant 0\}$, we have

$$\begin{aligned} |e^{-ps} - e^{-(p-1)s}| &= e^{-(p-1)\sigma} |\, e^s - 1| \\ &\leqslant e^{\mu(p-1)D^{\cdot}} (e^{2\mu D^{\cdot}} + 1 - 2\,e^{\mu D^{\cdot}} \cos \pi D^{\cdot})^{1/2} \\ &= e^{\mu(1-D^{\cdot})} (e^{2\mu D^{\cdot}} - 2\,e^{\mu D^{\cdot}} \cos \pi D^{\cdot} + 1)^{1/2}, \end{aligned}$$

that is, by (VI.5),

$$\begin{aligned} |e^{-ps} - e^{-(p-1)s}|^2 &\leqslant e^{2\mu}(e^{-2\mu D^{\cdot}} - 2\,e^{-\mu D^{\cdot}} \cos \pi D^{\cdot} + 1) \\ &< e^{2\mu}(\mu^2 + \pi^2) D^{\cdot 2}, \end{aligned}$$

and if μ is a positive root of (VI.3), we have

$$\left| \frac{e^{-ps} - e^{-(p-1)s}}{2} \right| < 1.$$

By (VI.2) the function is therefore holomorphic on S.

We set:

$$\mu = -\log D^{\cdot} - \log(-\log D^{\cdot}) + \log 2 + c(D^{\cdot}),$$

and show that $c = c(D^{\cdot}) = o(1)$ $(D^{\cdot} \to 0)$. In fact, we have:

$$\frac{4}{D^{\cdot 2}} = e^{2\mu}(\mu^2 + \pi^2)$$

$$= \frac{4e^{2c}}{D^{\cdot 2}(\log D^{\cdot})^2}[-\log D^{\cdot} - \log(-\log D^{\cdot}) + \log 2 + c]^2 \frac{\mu^2 + \pi^2}{\mu^2} \sim \frac{4\,e^{2c}}{D^{\cdot 2}},$$

which shows that $c(D^{\cdot}) = o(1)$. The theorem is now proved.

VI.3. Entire functions represented by Dirichlet series

In this section we assume that $\sigma_{a,f} = -\infty$ and, as above, that $D^{\cdot} < \infty$. Then $f(s) = (A, \Lambda)$ is an entire function with

$$\lim \frac{\log |a_n|}{\lambda_n} = -\infty.$$

By the classical theorem of Julia we know that every entire function other than a polynomial has the following property: there exists a ray through the origin such that in every angle around this ray the function takes all values, except possibly one, infinitely often. Such a ray is said to be a *Julia line*.

Translated into the language of Taylor-D series, the theorem of Julia can be stated in the following form: if $f(s) = \sum a_n e^{-ns}$ is an entire function and not a polynomial (in e^{-s}), then there exists a t_0 (in every interval $[2k\pi, 2(k+1)\pi[$) such that for every $\varepsilon > 0$ the function $f(s)$ takes in the strip $S(t_0, \varepsilon)$ all values, except possibly one, infinitely often.

Let us introduce the notion of such a line for every entire function $f(s)$ represented by a series (A, Λ). A line $t = t_0$ is said to be *J^*-line for f* if for every $\varepsilon > 0$ the function $f(s)$ takes in $S(t_0, \varepsilon)$ all the values, except possibly one, infinitely often. If f is not identically zero (we recall that we assume throughout $\lambda_1 > 0$), then it has a J-line (a classical Julia-line), but it can also have J^*-lines. It is easily seen that if $t = t_0$ is a J^*-line, then the half-line $\sigma < 0$, $t = 0$, is a J-line. The function $f(s) = e^{-s}$ has $\sigma = 0$, $t > 0$

(and $t<0$) as a J-line, but it has no J^*-line. Besides, no series (A, Λ) in which only finitely many a_n are different from zero has a J^*-line.

Theorems III.2.6 and III.2.9 lead to the following theorems.

Theorem VI.3.1. *Suppose that the step of* Λ *is positive and that* $f(s)=(A, \Lambda)$ *is an entire function whose* (R)*-order is infinite. Then* f *has a* J^**-line in every strip* $S(t_0, \pi a)$ *with* $a>\bar{D}^{\cdot}$ *and any real* t_0.

Theorem VI.3.2. *Let* $f(s)=(A, \Lambda)$ *be an entire function. If there exists a* t_0 *and* $a>\bar{D}^{\cdot}$ *such that the order of* f *in* $S(t_0, \pi a)$ *is infinite then for every* $d>a$, f *has in* $S(t_0, \pi d)$ *a* J^**-line.*

Theorem VI.3.3. *Suppose that the step of* Λ *is positive and that* $f(s)=(A, \Lambda)$ *is an entire function whose* (R)*-order is* $\rho>0$. *Then* f *has a* J^**-line in every strip* $S(t_0, \pi a)$ *with* $a>\max(\bar{D}^{\cdot}, 1/2\,\rho)$ *and any real* t_0.

Note that in VI.3.2 we do not postulate that the step should be positive. If we add this condition, then Theorem VI.3.2 only becomes a special case of Theorem VI.3.1.

Theorem VI.3.1 is a special case of Theorem VI.3.3. Let us first prove Theorem VI.3.2.

Let $\mathfrak{D}$ be the square $|\sigma-\sigma_0|<\pi d$, $|t-t_0|<\pi d$, where σ_0 is an arbitrary fixed real number, and let us consider in $\mathfrak{D}$ the family of holomorphic functions $F_c(s)=f(s+c)$, $s\in\mathfrak{D}$, $c\leqslant 0$. This is not a normal family. For if it were, only two cases could arise:

(1) There exists a sequence $\{c_n\}$ with $\lim c_n=-\infty$ such that for a $b(a<b<d)$

$$|F_{c_n}(s)| \leqslant M < \infty$$

on the square $\mathfrak{D}_b$ defined by $|\sigma-\sigma_0|\leqslant\pi b$, $|t-t_0|\leqslant\pi b$, which according to II.2 (where we set $R=b$) would imply that f is identically zero;

(2) $\lim F_c(s)=\infty$ as $\lim c=-\infty$, uniformly on $\mathfrak{D}_b$. This would contradict Theorem III.2.6.

The family $F_c(s)$ is therefore not normal in $\mathfrak{D}$ and consequently has there at least one irregular point $s'=\sigma'+it'$. And no matter what $\varepsilon>0$ is, $F_c(s)$ is not a normal family in $C(s', \varepsilon)$. This indicates that f takes in $S(t', \varepsilon)$ all values, except possibly one, infinitely often. Hence $t=t'$ is indeed a J^*-line.

Theorem VI.3.3 is proved in a completely similar manner, this time making use of Theorem III.2.9.

VI.4. Some applications of the composition theorems

Before stating the theorems of this section we have to prove the classical theorem of Cramèr. We also state the extension of this theorem due to Pólya (which is called the Cramèr-Pólya theorem) as well as an important contribution to this theorem by V. Bernstein, although only Cramèr's original theorem will really be used.

THEOREM VI.4.1. *Let* $f(s)=(A, \Lambda)$ *be uniform with* σ_a $(=\sigma_{a,f})$ *finite, and let* $\varphi(z)$ *be an entire function of exponential type* δ. *Let* S_f *be the singular set of* f *and* $S_{f,\delta}$ *the union of the disks of radius* δ *around the points of* S_f. *Let* Δ *be the part of the complement of* $S_{f,\delta}$ *with respect to the s-plane that is connected with the half-plane* $P_{\sigma'}$, $\sigma'>\sigma_a+\delta$. *Then the function* $F(s)=(C, \Lambda)$, *where* $C=\{c_n\}$, $c_n=\varphi(\lambda_n)a_n$, *is holomorphic in* Δ.

Let $\Phi(z)$ be the Laplace transform of $\varphi(z)$:

$$\Phi(z) = \int_0^\infty \varphi(u)\, e^{-uz}\, du.$$

It is known that $\Phi(z)$ is holomorphic for $|z|>\delta$ and that for every $\varepsilon>0$:

$$\text{(VI.6)} \qquad \varphi(z) = \frac{1}{2\pi i}\int_{C_\varepsilon} e^{zw}\Phi(w)\, dw,$$

where C_ε is the circle $|z|=\delta+\varepsilon$. This can be seen immediately by taking account of the fact that

$$\varphi(z) = \sum_{n\geqslant 0} d_n z^n$$

implies

$$\Phi(z) = \sum_{n\geqslant 0} \frac{n!\,d_n}{z^{n+1}} \quad \text{for} \quad |z| > \delta.$$

By writing down (A, Λ) for the variable $s-w$ and making use of (VI.6) we see that for sufficiently large σ

$$\text{(VI.7)} \qquad F(s) = \frac{1}{2\pi i}\int_{C_\varepsilon} f(s-w)\Phi(w)\, dw.$$

From this it follows that for $s \in \Lambda$ we can extend $F(s)$ on every Jordan curve in Δ (having one of its extremities in the half-plane $\sigma>\sigma'$ where σ' is sufficiently large), provided that ε is sufficiently small. This proves the theorem.

Now let $h(\theta)$ $(0 \leqslant \theta \leqslant 2\pi)$ by *the type of* φ *along the ray* $r\,e^{i\theta}$ $r>0$; that is,

$$h(\theta) = \limsup_{r \to \infty} \frac{\log |\varphi(re^{i\theta})|}{r}.$$

The *conjugate diagram* of φ is defined as the closed domain $\mathfrak{D}$ whose points $z=x+iy$ satisfy the inequality

$$x \cos \theta - y \sin \theta \leqslant h(\theta).$$

According to Pólya $\mathfrak{D}$ is the smallest convex domain in the exterior of which $\Phi(z)$ is holomorphic. Consequently we have

$$\mathfrak{D} \subset \overline{C(0, \delta)}.$$

Thus, denoting by $\mathfrak{D}_\varepsilon$ $(\varepsilon > 0)$ the union of the disks $C(z, \varepsilon)$, where z is an arbitrary point of $\mathfrak{D}$ and by L_ε the boundary of $\mathfrak{D}_\varepsilon$, we can also write by virtue of (VI.7):

$$F(s) = \frac{1}{2\pi i} \int_{L_\varepsilon} f(s - w)\Phi(w)\, dw.$$

From this we easily deduce the following theorem.

Theorem VI.4.2. *If* f, φ *and* F *are defined as in* VI.4.1 *and if* S_F *denotes the singular set of* F *and* $\mathfrak{D}$ *the conjugate diagram of* φ, *then*

$$S_F \subset S_f + \mathfrak{D}.$$

By using the method of Chapter V, that is, by dividing the plane (or rather part of the plane) suitably in accordance with the singularities of Φ and by starting out from the equation (VI.7), which is valid for sufficiently large σ, only to replace it later, by virtue of Cauchy's theorem, by an integral concerning the same function, but extended over a set of contours corresponding to the division, we obtain the following statement, which is due to V. Bernstein.

Theorem VI.4.3. *If the functions* f, φ *and* F *are defined as in* VI.4.2, Φ *being the Laplace transform of* φ *and* S_Φ *the singular set of* Φ, *then*

$$S_F \subset S_f + S_\Phi.$$

Theorems V.1.1 and VI.4.1 allow us to state the following theorem which concerns complementary series.

Theorem VI.4.4. *Let* $P=\{p_n\}$ *and* $Q=\{q_n\}$ *be two disjoint subsequences of*

Λ ($P \cup Q = \Lambda$, $P \cap Q = \phi$); let $\varphi(z)$ be an entire function of exponential type δ.

A and B being two complex sequences, we denote by $\varphi(\Lambda)$, $\varphi(P)$ and $A\varphi(P)$ the sequences $\{\varphi(\lambda_n)\}$, $\{\varphi(p_n)\}$, $\{a_n\varphi(p_n)\}$, and we set

$$\begin{aligned} f(s) &= (A, P) \\ \psi(s) &= (A\varphi(P), P) \\ \vartheta(s) &= (\varphi(\Lambda), \Lambda) - (B, Q). \end{aligned}$$

Suppose that f and ϑ are uniform and bounded, except at the singularities, in every half-plane $P_{\sigma'}$.

Then the function ψ is also uniform and the singular sets S_f, S_ψ and S_ϑ are linked in the following manner:

(a) *for every $\delta' > \delta$ every disk $C(s, \delta')$ with $s \in S_\psi$ contains a point of S_f,*

(b) $S_\psi \subset \overline{S_f + S_\vartheta}$.

In particular, if φ is an entire function of minimum type ($\delta = 0$), *then the assertion* (a) *can be replaced by the following:*

(c) $S_\psi \subset S_f$;

and the conclusions on the singular sets can be combined in this case by:

(d) $S_\psi \subset (\overline{S_f + S_\vartheta}) \cap S_f$.

We remark that if f and ϑ are not assumed to be uniform, the statement still remains valid, provided that we denote (as we have said at the end of Chapter V for the composition theorems) by S_f and S_ϑ the singular set of f and ϑ to which we add the cuts which make these functions uniform at the exterior of the sets so obtained. But evidently we can no longer state that ψ is a uniform function.

To prove the theorem it suffices to observe that as a consequence of the fact that P and Q are two disjoint parts of Λ, if we set

$$f(s) = (A, P) = (C, \Lambda), \quad \psi(s) = (D, \Lambda), \quad C = \{c_n\}, \quad D = \{d_n\},$$

we have $c_n = a_m$, $d_n = a_m\varphi(p_m)$ for $\lambda_n = p_m$; and $c_n = 0$ for $\lambda_n = q_k$. Theorem V.1.1 then provides the assertion (b). The assertion (a) follows from Theorem VI.4.1.

Here is a particular case of VI.4.4, stated for the Taylor series ($\Lambda = N$, $e^{-s} = z$).

Theorem VI.4.5. *Let* $P=\{p_n\}$, $Q=\{q_n\}$ *be two disjoint ascending sequences of positive integers. If there exists a series*

(VI.8) $\sum c_n z^{q_n}$

of finite radius of convergence having a pole among its singularities on the circle of convergence, then every series

(VI.9) $\sum d_n z^{p_n}$

of finite radius of convergence has at least two singular points on its circle of convergence.

If α is a pole of the series (VI.8) on its circle of convergence, and if R is the radius of convergence of (VI.9), we set

$$(B, Q) = \sum \alpha^{q_n} c_n\, e^{-q_n s} = \sum b_n\, e^{-q_n s}$$
$$(A, P) = \sum R^{p_n} d_n\, e^{-p_n s} = \sum a_n\, e^{-p_n s}.$$

Now clearly there exists a polynomial $\varphi(z)$ (hence a function of minimum type) such that the series

$$\sum \varphi(n) z^n$$

represents the principal part of the pole at 1 of the function (B, Q), with e^{-s} replaced by z. The function $\vartheta(s)=(\varphi(\Lambda), \Lambda)-(B, Q)$ therefore has all the points $2k\pi i$ (where k is an arbitrary integer) as regular points (here $\Lambda=N$). If (A, P) had on its axis of convergence $\sigma=0$ only one singular point (mod $2\pi i$), for example, the point $it_0 (\text{mod } 2\pi i)$, then we would have $it_0 \notin S_f + S_\vartheta$, which would be a contradiction to Theorem VI.4.4. This proves Theorem VI.4.5.

We remark that Theorem VI.4.5 can be proved more directly, first of all by taking account of the fact that if $\varphi(z)$ is a polynomial, then $(A\varphi(P), P)$ $(P \subset N)$ has no singularities other than those of (A, P), and subsequently by making use of Hadamard's classical composition theorem for the Taylor series (or rather, in the notation we have used here, the Taylor-D series (A, P) and $(\varphi(N), N)-(B, Q)$, $Q \subset N$, $Q \cap P=\phi$.

Theorem VI.4.5 has been generalized in different ways, but all the generalizations are concerned with Taylor series. For example, the word 'pole' has been replaced by the words 'algebraic-logarithmic singularity'.

We are now going to apply the arguments of (V.5). In fact, we shall only use the classical theorem of Hurwitz on Taylor series to end up with a theorem which itself only concerns Taylor series.

THEOREM VI.4.6. *Let* $A=\{a_n\}$ *be a sequence of rational numbers such that there exists an integer* N *with the property that all the quantities* a_nN^n $(n\geqslant 1)$ *are integers.*

Suppose that the function $F(z)=\sum a_nz^n$ *is regular at infinity and holomorphic for*

$$|z-N^2/(N^2-1)|\geqslant N/(N^2-1),$$

if $N\geqslant 2$; *and holomorphic for* $x\leqslant\frac{1}{2}$ $(z=x+iy)$ *if* $N=1$.

Then:

$$F(z)=\frac{P(z)}{(1-z)^h}\tag{VI.10}$$

where $P(z)$ *is a polynomial and* h *a positive integer.*

We set

$$f(z)=\sum\frac{a_nN^n}{z^n}\tag{VI.11}$$

$$\varphi(z)=\sum\frac{(-1)^{n-1}N^{n-1}}{z^n}.$$

The singularities of f are obtained from those of F by the transformation N/z. Hence all the singularities of f are in the disk $C(N, 1)$.

The theorem of Hurwitz (see (V.5)) applied to the series (VI.11) permits us to state, given that φ has only one singularity at $-N$, that the function

$$\psi(z)=\sum\frac{\gamma_{n-1}}{z^n},$$

with

$$\begin{aligned}\gamma_n&=(a_1N)N^n-\binom{n}{1}(a_2N^2)N^{n-1}+\cdots+(a_{n+1}N^{n+1})\\&=\delta N^{n+1}\Delta_na_1\end{aligned}$$

where Δ_na_1 denotes the n-th difference of the sequence $\{a_n\}$ with respect to a_1, and where δ is equal to $+1$ or -1, is holomorphic for $|z|\geqslant 1$.

Consequently we have

$$\limsup|\gamma_n|^{1/n}=\theta<1.$$

But since the γ_n (by assumption) are integers, we see that $\gamma_n=0$ for $n>n_0$.

Hence it follows from the identity

$$\sum a_n \frac{y^{n-1}}{(1+y)^n} = \sum_{p \geqslant 0} y^p \sum_{n \leqslant p+1} a_n \binom{p}{n-1} (-1)^{p-n+1}$$

that

$$\sum a_n z^{n-1} = \sum_{0 \leqslant p \leqslant n_0} \left(\frac{z}{1-z}\right)^p \frac{1}{1-z} (-1)^p \Delta_p a_1 = \frac{P_1(z)}{(1-z)^h},$$

where $P_1(z)$ is a polynomial. This proves the theorem.

Now we make the following remark. If the positive integer N is such that the quantities $a_n N^n$ $(n \geqslant 1)$ are integers, and if we denote by ρ the greatest distance between the point 1 and the singularities of F (see VI.4.6), we must have

$$N \geqslant \frac{1}{\rho} - 1,$$

provided that $F(z) = \sum a_n z^n$ is not of the form (VI.10).

So we obtain a lower bound for the integer N for a function represented by a series with rational coefficients to be, for example, algebraic, but not rational.

We consider again the function $F(z)$ whose Taylor coefficients are integers, of non-zero radius of convergence, the function being uniform, having infinity as a regular point or as a pole. Let S be the complement of the domain of existence of F in the open complex plane. According to our preceding definitions, S is therefore the singular set of F (of which we remove the point infinity if it is a pole). If E is a set in $\mathbf{R}^2$, we set $H = \overline{E \cap S}$. If H is finite and only contains poles of F at algebraic points, then there exists a polynomial $P(z)$ with integer coefficients such that the function $F(z)P(z)$ is regular at the points of H. If we now denote by $\Pi(F, E)$ the set of such polynomials, we define the *polar algebraic index of E with respect to F* as the smallest degree of the polynomials in $\Pi(F, E)$; we denote this index by $p(F, E)$.

If there is no polynomial P with integer coefficients such that $F(z)P(z)$ is a regular function at the points of H, that is, if $\Pi(F, E) = \phi$, we say that the *polar algebraic index $p(F, E)$ of E with respect to F is* $+\infty$. Thus, $p(F, E) = \infty$ in one of the following two cases (a) and (b).

(a) The set H is infinite.

(b) The set H is finite but contains at least one point of S which is not a pole at an algebraic point; in other words, E contains either a point of S which is not a pole, or a pole at a transcendental point.

The following theorem is an immediate consequence of Theorem VI.4.6 (corresponding to $N=1$).

THEOREM VI.4.7. *If the analytic continuation of* $F(z)=\sum a_n z^n$, *with integer coefficients, is a uniform function having infinity as a regular point or as a pole, then the polar algebraic index with respect to F of the (closed) half-plane* $x \leqslant \frac{1}{2}$ $(z=x+iy)$ *is not less than the polar index of the (open) half-plane* $x > \frac{1}{2}$ *with the point* $z=1$ *excluded. That is,*

$$\text{(VI.12)} \quad p(F, \{z | x \leqslant \tfrac{1}{2}\}) \geqslant p(F, \{z | x > \tfrac{1}{2}, z \neq 1\}).$$

More generally,

$$\text{(VI.13)} \quad \Pi(F, \{z | x \leqslant \tfrac{1}{2}\}) \subset \Pi(F, \{z | x > \tfrac{1}{2}, z \neq 1\}).$$

The relations (VI.12) *and* (VI.13) *remain valid if in the terms on the left of these relations the half-plane* $x \leqslant \frac{1}{2}$ *is replaced by the same half-plane with the poles of F at rational points excluded.*

Suppose that the relation (VI.13) does not hold when the half-plane $x \leqslant \frac{1}{2}$ is replaced by the same half-plane from which the poles at rational points, say $p_1/q_1, p_2/q_2, \ldots, p_m/q_m$, where the p_j and q_j are integers, are removed. Then there would exist a polynomial with integer coefficients $P(z)$ of degree less than $p(F, \{z | x > \frac{1}{2}, z \neq 1\})$, such that $F(z)P(z)$ has in the half-plane $x \leqslant \frac{1}{2}$ as singularities of finite distance (from the origin) only the poles at the places p_j/q_j $(j=1, 2, \ldots, m)$; the same product having in the half-plane $x > \frac{1}{2}$ singularities other than the point 1 at finite distance. On the other hand, there would exist a polynomial $Q(z)$ with integer coefficients, of the form

$$(q_1 z - p_1)^{\alpha_1}(q_2 z - p_2)^{\alpha_2} \ldots (q_m z - p_m)^{\alpha_m},$$

where $\alpha_1, \alpha_2, \ldots, \alpha_m$ are positive integers, $\Psi(z)=F(z)P(z)Q(z)$ is holomorphic on every compact set in $x \leqslant \frac{1}{2}$, and where the product has in $x > \frac{1}{2}$ the same singularities as $F(z)P(z)$. It is clear that there would also exist a polynomial (with integer coefficients) $R(z)$ of degree k such that $(\Psi(z)-R(z))z^{-k-1}$ is a Taylor series with integer coefficients representing a function regular at infinity, holomorphic for $x \leqslant \frac{1}{2}$ and having in the half-plane $x > \frac{1}{2}$ singularities other than the point 1; but this would be a contradiction to Theorem VI.4.6.

Here is an immediate corollary to Theorem VI.4.7.

THEOREM VI.4.8. *If the non-negative integers* a_n *are such that* $\limsup a_n^{1/n}=a$ *with* $1<a<2$, *and if the analytic continuation* F *of* $\sum a_n z^n$ *is a uniform function, with the point at infinity being regular or a pole, then* F *has apart from the singular point* $z=a^{-1}$ *another singular point at finite distance situated in the half-plane* $x\leqslant\frac{1}{2}$.

NOTES

[1] It is easy to see that $\sigma_{a,f}=0$ implies that $\sigma_1\leqslant 0$.

[2] To simplify the formulae we have preferred in this case the function $\mathfrak{N}(x)$ to $N(x)$, which denotes the number of λ_n *smaller than* x.

CHAPTER VII

ON THE BEHAVIOUR OF THE REMAINDERS OF A DIRICHLET SERIES IN THE DOMAIN OF EXISTENCE OF THE FUNCTION. APPLICATIONS

VII.1. Evaluation of the remainders of (A, Λ)

In this chapter we assume that Λ has a positive step h, and we set

$$\inf (\lambda_{n+1} - \lambda_n) = p.$$

Then $0<p\leqslant h$. To simplify the statements we also assume that the abscissa of convergence is equal to zero, that is,

$$\sigma_a = \sigma_c = \limsup \frac{\log |a_n|}{\lambda_n} = 0.$$

Let $C=\{c_n\}$ be a real sequence in which infinitely many terms may be equal to $-\infty$, but infinitely many are finite. Suppose that

$$\limsup \frac{c_n}{\lambda_n} = 0.$$

Then there exists a *concave function* $C(x)$ $(x\geqslant\lambda_1)$ having the following properties:

(1) $C(\lambda_n)\geqslant c_n$ $(n\geqslant 1)$

(2) $C(x)\geqslant 0$,

(3) Every other concave function $C_1(x)$ $(x\geqslant\lambda_1)$ satisfying the same inequalities also satisfies $C_1(x)\geqslant C(x)$. The function $C(x)$ is called *the smallest concave non-negative envelope of the sequence of points* $P_n=(\lambda_n, c_n)$ $(n\geqslant 1)$.

It is easily seen that $\lim C(x)/x=0$.

Given a series $f=(A, \Lambda)$ we denote by f_n the function represented by the sum

$$\sum_{m>n} a_m e^{-\lambda_m s},$$

that is, by the n-th remainder of (A, Λ).

We prove the following theorem:

THEOREM VII.1.1. *Suppose that* $f=(A, \Lambda)$ *is holomorphic in a simply connected domain* D *containing the half-plane of convergence* $(\sigma>0)$. *Let*

$C(x)$ be the smallest non-negative concave envelope of the sequence of points $(\lambda_n, \log|a_n|)$ $(n \geqslant 1)$.

For every compact domain $\bar{\Delta}$ contained in D there exists a constant $P = P(\Delta)$ such that when we set $q_n = e^{C(\lambda_n)}$, we have for $s \in \bar{\Delta}$:

$$|f_n(s)|\, e^{\lambda_n \sigma} \leqslant P q_n.$$

We set

$$\text{(VII.1)} \quad \eta_n = \frac{C(\lambda_{n+1}) - C(\lambda_n)}{\lambda_{n+1} - \lambda_n}.$$

We see immediately that η_n decreases to zero. We consider the function

$$\varphi_n(s) = f_n(s + \eta_n)\, e^{\lambda_n(s+\eta_n)} q_n^{-1}.$$

For $s \in \Delta$ and sufficiently large n we have:

$$\begin{aligned} |\varphi_n(s)| &= \Big|f(s + \eta_n)\, e^{\lambda_n(s+\eta_n)} q_n^{-1} - \sum_{m \leqslant n} a_m q_n^{-1}\, e^{-(\lambda_m - \lambda_n)(s+\eta_n)}\Big| \\ &\leqslant M_n\, e^{\lambda_n(\eta_n + \sigma)} q_n^{-1} \sum_{m \leqslant n} q_m q_n^{-1}\, e^{-(\lambda_m - \lambda_n)(\sigma + \eta_n)} \\ &= I_n(s) + J_n(s), \end{aligned}$$

where $M_n = \max|f(s+\eta_n)|$ as long as $s \in \bar{\Delta}$, $n \geqslant n_0$.

Evidently $M_n \leqslant M < \infty$ for $n \geqslant n_0$.

Taking account of (VII.1) and the fact that $C(x)$ is concave, we see that for $1 \leqslant m < n$ $(n > 1)$

$$\text{(VII.2)} \quad \frac{C(\lambda_n) - C(\lambda_m)}{\lambda_n - \lambda_m} \geqslant \eta_n.$$

Hence, if $\sigma < 0$, then by making use of (VII.2) for $m = 1$ we have:

$$\text{(VII.3)} \quad I_n(s) \leqslant M\, e^{\lambda_n \eta_n}\, e^{-C(\lambda_n)} \leqslant M\, e^{\lambda_1 \eta_1}.$$

From the same inequality (VII.2) (and still for $\sigma < 0$) we deduce:

$$\begin{aligned} \text{(VII.4)} \quad J_n(s) &= \sum_{m \leqslant n} e^{-C(\lambda_n) + C(\lambda_m) + (\lambda_n - \lambda_m)\eta_n}\, e^{(\lambda_n - \lambda_m)\sigma} \\ &\leqslant \sum_{m \leqslant n} e^{(\lambda_n - \lambda_m)\sigma} \leqslant \sum e^{np\sigma} \leqslant \frac{1}{1 - e^{p\sigma}}, \end{aligned}$$

where p is the quantity defined at the beginning of this chapter.

The inequalities (VII.3) and (VII.4) give for $\sigma < 0$, $s \in \bar{\Delta}$:

$$\text{(VII.5)} \quad |\varphi_n(s)| \leqslant \frac{K}{1 - e^{p\sigma}},$$

where K is a constant independent of n, $n \geqslant n_0$.

For $s \in \bar{\Delta}$, $\sigma > 0$ we have:

$$|\varphi_n(s)| = \Big|\sum_{m>n} a_m q_n^{-1} e^{-(\lambda_m-\lambda_n)(s+\eta_n)}\Big|$$
$$\leqslant \sum_{m>n} e^{C(\lambda_m)-C(\lambda_n)-(\lambda_m-\lambda_n)\eta_n} e^{(\lambda_m-\lambda_n)\sigma}$$

and according to (VII.1):

$$|\varphi_n(s)| = \sum_{m>n} e^{-(\lambda_m-\lambda_n)\sigma} \leqslant \sum e^{-np\sigma} \leqslant \frac{1}{1-e^{-p\sigma}}.$$

Thus, both for $\sigma > 0$ and $\sigma < 0$, $s \in \bar{\Delta}$ $(n \geqslant n_0)$:

$$\text{(VII.6)} \quad |\varphi_n(s)| \leqslant \frac{K_1}{|1-e^{-p\sigma}|},$$

where K_1 is a constant: $K_1 = K_1(\Delta)$.

We denote by $R[s_0, a]$ the square defined by $|\sigma-\sigma_0| < a$, $|t-t_0| < a$ $(s_0 = \sigma_0 + it_0)$, and we consider the function

$$\psi_n(s) = [1 - e^{-p(s-i(t_0-2\delta))}][1 - e^{-p(s-i(t_0+2\delta))}]\varphi_n(s).$$

According to (VII.6), f (and $\psi_n(s)$) is holomorphic in $R[it_0, 3\delta]$ for sufficiently small $\delta > 0$. Now $\psi_n(s)$ satisfies on the contour of $R[it_0, 2\delta]$ the inequality

$$|\psi_n(s)| \leqslant B < \infty \quad (n \geqslant 1),$$

where B is independent of n $(n \geqslant n_0)$.

Since on the contour of $R[it_0, \delta]$ for sufficiently small δ

$$\mu = \min|1 - e^{-p(s-i(t_0-2\delta))}| = \min|1 - e^{-p(s-i(t_0+2\delta))}| > 0,$$

we obtain for $s \in \overline{R[it_0, \delta]}$:

$$|\varphi_n(s)| \leqslant L\mu^{-2} \quad (n \geqslant 1),$$

where L is a constant.

Thus, for every $s_0 \in \bar{\Delta}$ there exists a square with this point at the centre such that on the square $\max|\varphi_n(s)| < M_0$, where M_0 does not depend on s_0. By the Borel-Lebesgue theorem, the function $\varphi_n(s)$ is bounded on $\bar{\Delta}$ by a quantity independent of n. This completes the proof.

We make the following remark concerning the smallest non-negative concave envelope of a sequence $\{c_n\}$. If $\limsup c_n = \infty$, then there exists a sequence $\{n_j\}$ such that $C(\lambda_{n_j}) = c_{n_j}$ $(j \geqslant 1)$, where the points (λ_{n_j}, c_{n_j}) are the vertices of the polygonal line defined by $y = C(x)$ $(x \geqslant \lambda_1)$. If $\limsup c_n = l$, with $-\infty < l < \infty$, then to every sequence $\{n_j\}$ for which

$$\liminf c_{n_j} > -\infty$$

there corresponds a constant c such that

$$\text{(VII.7)} \quad c_{n_j} = C(\lambda_{n_j}) \leqslant c_{n_j} + c.$$

A sequence $\{n_j\}$ satisfying (VII.7) is called *sequence of principal indices for* $\{c_n\}$.

Theorem VII.1.1 leads to the following proposition:

VII.1.2. *Let* $\{n_j\}$ *be a sequence of principal indices for* $\{\log |a_n|\}$.

Suppose that $f = (A, \Lambda)$ *is holomorphic in a simply-connected domain* D *containing the half-plane* $\sigma > 0$.

For every compact domain $\bar{\Delta}$ *contained in* D, *every* $s \in \bar{\Delta}$ *and every* $j \geqslant 1$ *we have:*

$$|f_{n_j}(s)|\, e^{\lambda_{n_j}\sigma} \leqslant Q |a_{n_j}|,$$

where Q *is a constant.*

The following elementary lemma is also useful:

VII.1.3. *If* $f = (A, \Lambda)$ *has a pole on its axis of convergence, then*

$$\text{(VII.8)} \quad \limsup |a_n| > 0.$$

If (VII.8) would not hold, then to every $\varepsilon > 0$ there would correspond an n_0 such that $|a_n| < \varepsilon$ for $n > n_0$; and then for $\sigma > 0$:

$$\begin{aligned} |f(s)| &\leqslant \sum |a_n|\, e^{-\lambda_n\sigma} = \sum_{n \leqslant n_0} |a_n|\, e^{-\lambda_n\sigma} + \sum_{n > n_0} |a_n|\, e^{-\lambda_n\sigma} \\ &\leqslant A(\sigma) + \varepsilon \sum e^{-np\sigma} \leqslant A(\sigma) + \varepsilon(1 - e^{-p\sigma})^{-1}, \end{aligned}$$

where $p > 0$ is the constant used above.

$A(\sigma)$ is bounded for $\sigma \geqslant 0$. For $\sigma \downarrow 0$ we would also have:

$$f(\sigma + it) = o(\sigma^{-1})$$

(uniformly in t). No point it_0 on the axis of convergence could therefore be a pole, contrary to our assumption.

Theorem VII.1.1 (or its Corollary VII.1.2) enable us to introduce apart from a function $f=(A, \Lambda)$ an auxiliary function which has the effect of 'separating' the singularities of f on its axis of convergence from all its other singularities.

THEOREM VII.1.4. *Let $\{n_j\}$ be a sequence of principal indices of the sequence*

$$\{\log |a_n|\}(\{a_n\} = A).$$

Suppose that the axis of convergence ($\sigma=0$) is not a cut for (A, Λ), and let T be the domain composed of the half-planes $\sigma<0$, $\sigma>0$ and the set of regular points of f on $\sigma=0$.

From $\{n_j\}$ we can extract a sequence $\{p_k\}$ having the following property: the sequence

$$f_{p_k}(s)\, e^{\lambda_{p_k} s} a_{p_k}^{-1} \quad (k \geqslant 1)$$

tends, uniformly on every compact set contained in T, to a function $g(s)$ that is holomorphic in T.

There exist two sequences $M=\{\mu_n\}$ and $M'=\{\mu'_n\}$ such that

$$\mu_1 \geqslant h, \quad \mu_{n+1} - \mu_n \geqslant h \quad (n \geqslant 1)$$
$$\mu'_1 \geqslant h, \quad \mu'_{n+1} - \mu'_n \geqslant h \quad (n \geqslant 1),$$

where h is the step of Λ, and two sequences $B=\{b_n\}$, $B'=\{b'_n\}$ such that for $\sigma>0$, $\sigma<0$ we have, respectively,

$$g(s) = \sum b_n\, e^{-\mu_n s}$$
$$g(s) = \sum b'_n\, e^{\mu'_n s} - 1.$$

The function $g(s)$ has singular points on $\sigma=0$. They are all the singular points of $f(s)$.

Every isolated singular point of g is a simple pole.

We remark that the condition on the n_j to be principal indices of the sequences $\{\log |a_n|\}$, that is, that

$$\log |a_{n_j}| \leqslant C(\lambda_{n_j}) \leqslant \log |a_{n_j}| + c,$$

where $C(x)$ is the smallest non-negative concave envelope of the points $(\lambda_n, \log |a_n|)$, can be replaced by a more general condition. But the general theorem is more difficult to prove. The statement we have given here suffices for the applications we have in mind.

The sequence of functions $\{F_j(s)\}$ defined by

$$\text{(VII.9)}\quad F_j(s) = f_{n_j}(s)\, e^{\lambda_{n_j} s} |a_{n_j}|^{-1} \quad (j \geqslant 1)$$

constitutes a normal family which is bounded on the $\bar{\Delta}$ defined in VII.1.1. From it we can extract a sequence $F_{j_\nu}(s)$ that converges uniformly on every compact set contained in Δ to a holomorphic function, which we denote by $g(s)$.

From the fact that $C(x)$ is concave and from the relation

$$\lim C(\lambda_n)/\lambda_n = 0$$

it follows that

$$\lim [C(\lambda_{n+1}) - C(\lambda_n)] = 0;$$

in other words, if as in VII.1.1 we set $\log q_n = C(\lambda_n)$ we have:

$$\text{(VII.10)}\quad \lim q_{n+1}/q_n = 1.$$

We set $n_{j_\nu} = m_\nu$ $(\nu \geqslant 1)$. Taking account of the inequality $q_{m_\nu} \leqslant C|a_{m_\nu}|$ (where C is a constant) and of (VII.10) we obtain for a positive integer m:

$$\text{(VII.11)}\quad \left|\frac{a_{m_\nu \pm m}}{a_{m_\nu}}\right| \leqslant \frac{q_{m_\nu \pm m}}{|a_{m_\nu}|} \leqslant C\frac{q_{m_\nu \pm m}}{q_{m_\nu}}$$

(we consider $m_\nu - m$ only for $m < m_\nu$), that is:

$$\text{(VII.12)}\quad \limsup \left|\frac{a_{m_\nu \pm m}}{a_{m_\nu}}\right| \leqslant C.$$

Note that it is no loss of generality in the theorem if we assume that there exists a constant $H < \infty$ such that

$$\text{(VII.13)}\quad \lambda_{n+1} - \lambda_n \leqslant H.$$

In fact, if this condition does not hold, it is sufficient to replace, on the one hand, Λ by a sequence Λ' having the same step as Λ and containing it and, on the other hand, A by the sequence A' whose elements corresponding to the λ_n of Λ are those of (A, Λ) where the other elements of A' are all zero. We observe immediately that in proving the theorem for

(A', Λ') we also prove it for (A, Λ). Hence we assume that (VII.13) holds.

If h is the step of Λ, let $0<\varepsilon<h-\varepsilon$. For sufficiently large ν we have:

$$(\alpha)\ m(h-\varepsilon)\leqslant \lambda_{m_\nu+m}-\lambda_{m_\nu}\leqslant mH,$$
$$(\beta)\ m(h-\varepsilon)\leqslant \lambda_{m_\nu}-\lambda_{m_\nu-m}\leqslant mH,$$

where the inequality between the first and the second term of (β) only holds when $m_\nu-m\geqslant p=p(\varepsilon)$.

It follows that we can extract from $\{m_\nu\}$ a partial sequence $\{p_k\}$ (for example, by the diagonal process) having the following property: the limits

$$\lim\,(\lambda_{p_k+m}-\lambda_{p_k})=\mu_m, \qquad \lim a_{p_k+m}a_{p_k}^{-1}=b_m,$$
$$\lim\,(\lambda_{p_k}-\lambda_{p_k-m})=\mu'_m, \qquad \lim a_{p_k-m}a_{p_k}^{-1}=b'_m,$$

exist and satisfy the relations

$$\text{(VII.14)}\quad |b_m|\leqslant C, \qquad |b'_m|\leqslant C$$

$$\text{(VII.15)}\quad \mu_1\geqslant h, \qquad \mu_{m+1}-\mu_m\geqslant h,$$
$$\mu'_1\geqslant h, \qquad \mu'_{m+1}-\mu'_m\geqslant h.$$

For $\sigma>0$ and $\sigma<0$ we set, respectively:

$$\text{(VII.16)}\quad g^+(s)=\sum b_m\,e^{-\mu_m s},$$

$$\text{(VII.17)}\quad g^-(s)=\sum b'_m\,e^{\mu'_m s}.$$

According to (VII.14) and (VII.15) the series (VII.16) and (VII.17) converge, respectively, in the half-planes $\sigma>0$ and $\sigma<0$. And by the definition of $g(s)$ (which in Δ is the limit of the sequence F_{j_ν} $(n_{j_\nu}=m_\nu)$), for $\sigma>0$ we have:

$$g(s)=g^+(s).$$

If D is defined as in VII.1.1, we now suppose that the compact set $\bar{\Delta}$ contained in D forms part of the half-plane $\sigma<0$. Then we have uniformly on $\bar{\Delta}$

$$\lim f(s)\,e^{\lambda_{m_\nu}s}a_{m_\nu}^{-1}=0.$$

This allows us to write:

$$\text{(VII.18)}\quad \lim F_{j_\nu}(s)=-\lim\sum_{m<m_\nu}\frac{a_{m_\nu-m}}{a_{m_\nu}}\,e^{(\lambda_{m_\nu}-\lambda_{m_\nu-m})s}-1=g(s).$$

The definition of the sequences $\{\mu_n'\}$, $\{b_n'\}$ enables us therefore to write for $\sigma<0$:

$$g(s) = g^-(s) - 1.$$

Thus, $g(s)$ cannot have singularities outside the axis of convergence ($\sigma=0$); and its singularities are the same as that of $f(s)$. From (VII.14) and (VII.15) it follows that for $\sigma>0$:

$$|g(s)| = |g^+(s)| \leqslant C\sum e^{-mh\sigma} = Ce^{-h\sigma}(1-e^{-h\sigma})^{-1}.$$

A similar inequality is valid for $|g^-(s)|$ in the half-plane $\sigma<0$. Thus, for $\sigma\neq 0$, $|\sigma|\leqslant 1$

$$|g(s)| \leqslant K|\sigma|^{-1},$$

where K is constant.

Suppose now, contrary to our assertion, that $g(s)$ is regular on $\sigma=0$.

Let $0<a<1$ and t_0 be arbitrary real and consider on the square $R[it_0, 2a]$ ($|\sigma|\leqslant 2a$, $|t-t_0|\leqslant 2a$) the function

$$\text{(VII.19)}\quad G(s) = g(s)[s - i(t_0 + 2a)][s - i(t_0 - 2a)].$$

Then $G(s)$ would be holomorphic on this square, and on its sides we would have

$$|G(s)| \leqslant 20Ka^{-1}.$$

On the square $R[it_0, a]$ we would therefore have $|g(s)|\leqslant 20Ka^{-1}$.

In other words, the function $g(s)$ would then be an entire function and bounded in the whole plane; consequently, it would be a constant; but $g(s)$ tends to zero as σ tends to $+\infty$ and to -1 as σ tends to $-\infty$. Hence we have a contradiction. Thus, $g(s)$ necessarily has singularities on $\sigma=0$. By the definition of g, these singularities are those of f.

Let us show finally that every isolated singular point of g is a simple pole. Let $i\tau$ be such a point, and consider again the function $G(s)$ defined by (VII.19), where $a>0$ is such that the closed square $\overline{R[i\tau, 3a]}$ contains no singularities other than $i\tau$ and where $t_0=\tau\pm 3a$.

On $t_0\mp a=\tau\pm 2a$, $|\sigma|\leqslant a$, we find that $|g(s)|\leqslant 20Ka^{-1}$. Thus, on the sides of $R[i\tau, 2a]$ we have $|g(s)|\leqslant Pa^{-1}$ (for $|\sigma|=2a$ we have $|g(s)|\leqslant K(2a)^{-1}$), where P is constant. Thus, $i\tau$ can only be a simple pole.

The proof of the theorem is now complete.

VII.2. Applications of the results of Chapter III and of VI.1 to series meromorphic on an interval of the axis of convergence

We prove the following theorem.

THEOREM VII.2.1. *Let* $\bar{D}^{\cdot}$ *be the mean upper density of* Λ *and suppose that the sequences* Λ *(of step* h*) and* A *are such that there exist three constants* $H>0$, $C>0$, $N\geqslant 1$ *with*

$$\text{(VII.20)}\quad \lambda_{n+1} - \lambda_n < H \quad (n \geqslant 1)$$

$$\text{(VII.21)}\quad C^{-1} \leqslant \max_{1\leqslant \nu\leqslant N} |a_{n+\nu}| \leqslant C \quad (n \geqslant 1).$$

Suppose that $f(s)=(A, \Lambda)$ *does not have in a strip* S *defined by* $a<t<b$, *where* $b-a>2\pi(\bar{D}^{\cdot}+h^{-1})$, *singularities other than simple poles* $i\alpha_p$ $(p=1, 2, \ldots, q)$, *where the* α_p *are real and* $f(s)$ *remains bounded in* S *outside every neighbourhood of the points* $i\alpha_p$ $(p=1, 2, \ldots, q)$.

Then the function $f(s)$ *is holomorphic for* $\sigma<0$, *where it is given by a series*

$$f(s) = \sum a_n' e^{\lambda_n' s} + d. \quad (d \text{ a constant})$$

Furthermore:

$$\lambda_{n+1} - \lambda_n \geqslant h, \qquad \lambda'_{n+1} - \lambda'_n \geqslant h, \qquad |a_n'| \leqslant C.$$

Every isolated singularity $i\alpha$ *of* f *on* $\sigma=0$ *is a simple pole, and there exist* q *integers* $m_1, m_2, \ldots, m_q$ *such that*

$$\text{(VII.22)}\quad \alpha = m_1\alpha_1 + m_2\alpha_2 + \cdots + m_q\alpha_q.$$

In other words, the α_p $(p=1, 2, \ldots, q)$ form an 'integral basis' for all the α, which are the ordinates of the points of other isolated singularities of f on the axis of convergence.

We recall that the step h of Λ in this chapter is supposed to be positive and that the analytic continuation of (A, Λ) into S (from which the points $i\alpha_p$ $(p=1, 2, \ldots, q)$ are removed) is direct.

Remark. The conditions (VII.20) and (VII.21) follow from other conditions in the statement of Theorem VII.2.1, namely from the fact that f only has simple poles on the interval in question of the axis of convergence. But when the λ_n are not integers, the proof of this fact is not immediate.

It follows from (VII.20) and (VII.21) that there exists a sequence $\{n_j\}$ of principal indices of $\{\log |a_n|\}$ satisfying the conditions

(VII.23) $n_{j+1} - n_j \leqslant N$

(VII.24) $C_1^{-1} \leqslant |a_{n_j}| \leqslant q(n_j) = e^{C(\lambda_{n_j})} \leqslant C_1$,

where $C(x)$ is the smallest non-negative concave envelope of the set of points $P_n = (\lambda_n, \log |a_n|)$ $(n \geqslant 1)$, and C_1 is a constant such that $C(x) \leqslant \log C_1$ $(x \geqslant \lambda_1)$.

Consider the family of functions

$$f_x^*(s) = \Big[f(s) - \sum_{\lambda_n \leqslant x} a_n e^{-\lambda_n s}\Big] e^{xs} \quad (x > 0).$$

Let $j(x)$ be the largest integer j such that $\lambda_{n_j} \leqslant x$, and set $m(x) = n_{j(x)}$. Let $n(x)$ be the largest integer n such that $\lambda_n \leqslant x$. Then

$$0 \leqslant n(x) - m(x) \leqslant N.$$

If $n(x) \geqslant m(x) + 1$, we can write

$$f_x^*(s) = [f_{m(x)}(s) - (a_{m(x)+1} e^{-\lambda_{m(x)+1} s} + \cdots + a_{n(x)} e^{-\lambda_{n(x)} s})] e^{xs},$$

with $x - \lambda_{m(x)+1} \leqslant NH$; and if $n(x) = m(x)$, then

$$f_x^*(s) = f_{m(x)}(s)\, e^{xs}.$$

It follows from Theorems VII.1.4 that the family $f_x^*(s)$ $(x > 0)$ is bounded on any compact set containing no singularities of f and that we can extract from every sequence $\{x_k\}$, $x_k > 0$, $x_k \uparrow \infty$, a partial sequence $\{y_j\}$ such that $f_{y_j}^*(s)$ tends uniformly on this compact set to a function $G(s)$ having the following properties:

(1) For $\sigma > 0$ and $\sigma < 0$, respectively,

(VII.25) $G(s) = \sum d_n e^{-\nu_n s} + c,$

$$G(s) = \sum d_n' e^{\nu_n' s} + d,$$

where c and d are constants and where for $n \geqslant 1$:

(VII.26) $\nu_{n+1} - \nu_n \geqslant h$

$$\nu_{n+1}' - \nu_n' \geqslant h$$

$$|d_n| \leqslant C$$

$$|d_n'| \leqslant C$$

(we see easily that C is the constant occurring in (VII.21)).

(2) $G(s)$ is holomorphic outside the singularities of f on the axis of convergence $\sigma=0$.

(3) $G(s)$ has singularities on $\sigma=0$.

(4) The isolated singular points of $G(s)$ are simple poles.

(In fact, (3) follows from VII.1.4 as long as $d\neq c$, but we shall prove this statement directly in the subsequent analysis.)

Let us now show that every simple pole of f on $\sigma=0$ is also a simple pole for G. Let $i\beta$ be such a pole for f; we can write

$$f(s) = \frac{r}{s - i\beta} + \varphi(s),$$

where $\varphi(s)$ is a holomorphic function in a disk $C(i\beta, \delta)$ $(\delta>0)$. In this disk the functions $f^*_{y_j}(s)$ only have $i\beta$ as a singularity, namely as a simple pole with the residue $re^{i\beta y_j}$; the functions

$$\varphi_{y_j} = f^*_{y_j}(s) - \frac{re^{i\beta y_j}}{s - i\beta}$$

are therefore holomorphic in $C(i\beta, \delta)$. If we extract from $\{y_j\}$ a sequence $\{y'_j\}$ tending to infinity and such that $e^{i\beta y'_j}$ tends to a limit l, then the functions $\varphi_{y'_j}$ tend uniformly on the boundary of the disk, and consequently on the closed disk, to a limit that is holomorphic in the disk. The limit of $f^*_{y'_j}(s)$ which is $G(s)$, is therefore of the form

$$\text{(VII.27)}\quad G(s) = \frac{r'}{s - i\beta} + G_0(s),$$

where $r'=lr$, and $G_0(s)$ is holomorphic in $C(i\beta, \delta)$.

Since $G(s)$ is determined by the sequence $\{y_j\}$, the quantity l, and consequently the quantity $r'=rl(|l|=1)$, cannot vary with the choice of the sequence $\{y'_j\}\subset\{y_j\}$. It follows that $\lim e^{i\beta y_j}$ exists and is equal to l. That is, (VII.27) holds, with $r'=r\lim e^{i\beta y_j}$.

Hence if $\{y_j\}$ is a sequence for which $f^*_{y_j}(s)$ tends to $G(s)$ defined above, then $G(s)$ has the points $i\alpha_p$ $(p=1, 2, \ldots, q)$ as simple poles. If the residues of the poles $i\alpha_p$ of f are the quantities

$$r_p \quad (p = 1, 2, \ldots, q),$$

then the residues of the same poles of G are r_pl_p, where $l_p=\lim e^{i\alpha_p y_j}$ and these limits exist $p=1, 2, \ldots, q$.

Now by Dirichlet's approximation theorem we can choose a sequence y_j tending to ∞ with the property that

$$l_p = 1 \quad (p = 1, 2, \ldots, q),$$

that is, $\lim \alpha_p y_j = 0 \pmod{2\pi}$ for every

$$p = 1, 2, \ldots, q.$$

Thus, on $]ia, ib[$ of $\sigma = 0$ the function $F(s) = f(s) - G(s)$ is holomorphic. We denote by $\{\gamma_n\}$ the union $\{\lambda_n\} \cup \{\nu_n\}$ and set

$$\text{(VII.28)} \quad F(s) = \sum e_n\, e^{-\gamma_n s} = \sum a_n\, e^{-\lambda_n s} - \sum d_n\, e^{-\nu_n s}.$$

It is easy to see that if we denote by $\overline{D}^{\cdot}_{\gamma}$ the mean upper density of γ_n, then

$$\overline{D}^{\cdot}_{\gamma} \leqslant \overline{D}^{\cdot} + h^{-1}.$$

Thus, the function $F(s)$ given by (VII.28) is holomorphic and bounded in the strip S whose width exceeds $2\pi \overline{D}^{\cdot}_{\gamma}$. We conclude, by Theorem III.2.3, that $F(s)$ is identically zero, that is:

$$\text{(VII.29)} \quad f(s) = G(s) - c.$$

From this it follows, first of all, (by (4)) that the isolated singular points of f on $\sigma = 0$ are simple poles and that $f(s)$ is also holomorphic for $\sigma < 0$. Further, by (VII.25), (VII.26) and (VII.29) $\lambda_n = \nu_n$, hence:

$$\lambda_{n+1} - \lambda_n \geqslant h \quad (n \geqslant 1);$$

for $\sigma < 0$:

$$f(s) = \sum a'_n\, e^{\lambda'_n s} + d,$$

where $\lambda'_n = \nu'_n$, $a'_n = d'_n$; hence:

$$\lambda'_{n+1} - \lambda'_n \geqslant h \quad (n \geqslant 1)$$
$$|a'_n| < C.$$

It now remains to show that there exist q integers $m_1, m_2, \ldots, m_q$ such that (VII.22) holds.

We show that the contrary assertion leads to a contradiction.

If α is such that there exist integers $m, p_1, p_2, \ldots, p_q$ satisfying the relation

$$\text{(VII.30)} \quad m\alpha = p_1\alpha_1 + p_2\alpha_2 + \cdots + p_q\alpha_q,$$

then we denote by μ the smallest possible integer m having the property (VII.30). If (VII.22) were not possible, we would have $\mu > 1$, and every other integer m satisfying (VII.30) would be a multiple of μ. Setting $\beta = \mu^{-1}$ (<1), the numbers $\alpha, \beta, \alpha_1, \alpha_2, \ldots, \alpha_q$ would have the following property:

For arbitrary integers $n, n_1, n_2, \ldots, n_q$ for which

$$\text{(VII.31)} \quad n\alpha + n_1\alpha_1 + \cdots + n_q\alpha_q = 0,$$

the quantity $n\beta$ $(0 < \beta < 1)$ is an integer.

From a simple version of Kronecker's approximation theorem it would follow that for arbitrarily given $\varepsilon > 0$, $T > 0$ the systems of inequalities

$$\begin{aligned} \text{(VII.32)} \quad -\varepsilon < t\alpha - \beta < \varepsilon \quad & (\mathrm{mod}\ 1) \\ -\varepsilon < t\alpha_p < \varepsilon \quad & (\mathrm{mod}\ 1), \quad (p = 1, 2, \ldots, q), \end{aligned}$$

would have a solution $t > T$.

If (VII.31) could not be realized with any system of integers $n, n_1, n_2, \ldots, n_q$, then (VII.32) would equally have a solution $t > T$, and this for an arbitrary β with $0 < \beta < 1$.

In other words, if α could not be expressed as a linear combination with integral coefficients of the α_p $(p = 1, 2, \ldots, q)$, then there would exist a $0 < \beta < 1$ such that (VII.32) would have a solution $t > T$.

Hence there would exist an η_0 with $0 < \eta_0 < \frac{1}{2}$ such that to every $\varepsilon > 0$, $T > 0$ there would correspond a $t > T$ and integers $n, n_1, n_2, \ldots, n_q$ satisfying the inequalities

$$\begin{aligned} |t\alpha_p - n_p| < \varepsilon \quad & (p = 1, 2, \ldots, q) \\ \eta_0 < |t\alpha - n| < 1 - \eta_0. & \end{aligned}$$

And then we could associate with every sequence $\{\varepsilon_j\}$, $\varepsilon_j \downarrow 0$, a sequence $\{y_j\}$ and $q+1$ sequences $n_j^0, n_j^{(1)}, \ldots, n_j^{(q)}$ such that

$$\begin{aligned} \text{(VII.33)} \quad |y_j\alpha_p - 2\pi n_j^{(p)}| < \varepsilon_j \quad & (p = 1, 2, \ldots, q; j \geqslant 1), \\ 2\eta_0\pi < |y_j\alpha - 2\pi n_j^{(0)}| < 2(1 - \eta_0)\pi \quad & (j \geqslant 1). \end{aligned}$$

Choosing the function $G(s)$ with the y_j satisfying (VII.33) we would find, first of all, as above, that $G(s)$ has the poles $i\alpha_1, i\alpha_2, \ldots, i\alpha_q$ with the same residues as f; but if the residue of $i\alpha$ for f is r, say, the residue of this pole for G would be $r' = r\, e^{2\pi i\theta} \neq r$ (because $e^{2\pi i\theta} = \lim e^{i\alpha y_j}$, with $\eta_0 \leqslant \theta \leqslant 1 - \eta_0$), which is a contradiction. Thus, the theorem is proved.

CHAPTER VIII

APPLICATIONS TO THE GENERALIZED RIEMANN FUNCTIONAL EQUATION

VIII.1. Number of independent solutions. Links between exponents

The Riemann zeta function $\zeta(s)$, the analytic continuation of the Dirichlet series

$$\sum \frac{1}{n^s} = \sum e^{-\log ns},$$

which is known to play a fundamental role in the theory of numbers (for example, in the problem of the distribution of prime numbers), satisfies the so-called *Riemann functional equation*

$$\text{(VIII.1)}\quad \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

Hamburger has shown that if the series

$$\text{(VIII.2)}\quad f(s) = \sum \frac{a_n}{n^s}, \quad g(1-s) = \sum \frac{b_n}{n^{1-s}}$$

converge absolutely for $\sigma > 1$, and $\sigma < c < 0$, respectively, where in addition $f(s)$ is assumed to be of the form $G(s)/P(s)$, with $G(s)$ an entire function of finite order and $P(s)$ a polynomial, then the functional equation

$$\text{(VIII.3)}\quad \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)f(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)g(1-s)$$

implies that $f(s) = g(s) = a_1\zeta(s)$.

Thus, the Equation (VIII.3) characterizes in a way the function $\zeta(s)$, provided that f and g are represented by Dirichlet series whose sequence of exponents is $\{\log n\}$ (in the language we have used right through this book); true, certain simple conditions on the behaviour of these functions in the entire plane have to be added.

This is where the more general problem arises. As always, let $\Lambda=\{\lambda_n\}$, $M=\{\mu_n\}$ be two positive strictly increasing sequences and let

$$\text{(VIII.4)}\quad \varphi(s) = \sum \frac{a_n}{\lambda_n^s}, \quad \psi(s) = \sum \frac{b_n}{\mu_n^s}$$

be two series having abscissae of absolute convergence. Imitating the previous notation, we could denote such series by $(A, \log \Lambda)$, $(B, \log M)$, respectively, where φ and ψ denote not only the sums of the series but also their respective analytic continuations. If δ is a given positive number, suppose that the following relation holds:

$$\text{(VIII.5)}\quad \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\varphi(s) = \pi^{-(\delta-s)/2}\Gamma\left(\frac{\delta-s}{2}\right)\psi(\delta-s).$$

Clearly the behaviour of the function φ and ψ in the whole plane must be specified in a precise fashion.

Given the sequences $\{\lambda_n\}$ and $\{\mu_n\}$ and the quantity $\delta>0$, we can ask for the 'size' of the class of pairs (φ, ψ) satisfying (VIII.5) We can also ask to what extent we can choose the sequences $\{\lambda_n\}$ and $\{\mu_n\}$ themselves (in general or in relation to a given δ). These problems will be treated in the present chapter.

Now we lay down the explicit meaning which we give to the expression 'the pair (φ, ψ) satisfies the generalized Riemann equation (VIII.5)'.

If the sequences $\{\lambda_n\}$ and $\{\mu_n\}$ are defined as above, if $\delta>0$ is given, if φ and ψ are defined by (VIII.4), where the series have abscissae of absolute convergence, we suppose that there exists a function $\chi(s)$, not identically zero, holomorphic at finite distance outside a compact set K, and of exponential type zero in every vertical strip, that is,

$$\text{(VIII.6)}\quad \lim_{|t|\to\infty} \frac{\log|\chi(\sigma+it)|}{t} = 0,$$

uniformly in σ for $a\leqslant\sigma\leqslant b$, a and b being arbitrary; here χ is linked with φ and ψ by the following relations.

For sufficiently large $\sigma>0$:

$$\text{(VIII.7)}\quad \chi(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\varphi(s).$$

For sufficiently large $-\sigma>0$:

$$\text{(VIII.8)}\quad \chi(s) = \pi^{-(\delta-s)/2}\Gamma\left(\frac{\delta-s}{2}\right)\psi(\delta-s).$$

Then we say that the pair (φ, ψ) satisfies (VIII.5).

Now, as we have said above, since we are looking for conditions on $\Lambda=\{\lambda_n\}$, $M=\{\mu_n\}$ and $\delta>0$ under which there exist a_n and b_n $(n \geqslant 1)$, not all zero, which make (VIII.5) possible (or, in a more general manner, since we are looking for the number of such sequences $\{a_n\}$, $\{b_n\}$ from which all the others can be formed), we agree to say also that the pair (φ, ψ) satisfies the *Riemann functional equation corresponding to* (Λ, M, δ).

We begin with the following lemma:

VIII.1.1. *If the pair* (φ, ψ) *defined by* (VIII.4) *satisfies* (VIII.5) *and if we set*

$$\text{(VIII.9)}\quad g(s) = (B, 2\pi M) = \sum b_n\, e^{-2\pi\mu_n s},$$

then $\sigma_{a,g}=0$; *for* $\sigma>0$ *the following relation is satisfied for sufficiently large* d_0:

(VIII.10)

$$g(s) = K(s) + \pi^{-q}\Gamma(q) \sum a_n \left(\frac{s}{(s^2+\lambda_n^2)^q} + \sum_{0 \leqslant 2j \leqslant d_0 - 2q} A_j \frac{s^{1+2j}}{\lambda_n^{2q+2j}}\right),$$

where

$$\text{(VIII.11)}\quad A_j = \frac{(-1)^j\Gamma(q+j)}{j!\,\Gamma(q)} \qquad \left(q = \frac{\delta+1}{2}\right),$$

and where $K(s)$ *is an analytic function on the Riemann surface of* $\log s$ *with*

$$\text{(VIII.12)}\quad K(s) = O(|s|^m) \quad (|s| \to \infty,\ m \text{ a constant})$$

in every angle $|\arg s| \leqslant \theta_0$ $(\theta_0>0$ *positive,* m *depending on* $\theta_0)$.

For $\sigma>0$ and $d_0>0$ we have:

$$\text{(VIII.13)}\quad e^{-s} = \frac{1}{2\pi i}\int_{d_0-i\infty}^{d_0+i\infty} \Gamma(w)s^{-w}\,dw.$$

We also have:

$$\text{(VIII.14)}\quad \Gamma(w) = \pi^{-1/2}2^{w-1}\Gamma\left(\frac{w}{2}\right)\Gamma\left(\frac{w+1}{2}\right),$$

$$\text{(VIII.15)}\quad \frac{1}{2\pi i}\int_{d_0-i\infty}^{d_0+i\infty} z^{-w}\Gamma(p-w)\Gamma(w)\,dw = \Gamma(p)\Big[(1+z)^{-p} + \sum_{0 \leqslant j < d_0 - p} A_j z^{-p-1}\Big]$$

with

$$A_j = \frac{(-1)^j \Gamma(p+j)}{j!\,\Gamma(p)}, \quad \text{for} \quad |\arg z| < \pi,\ d_0 > p > 0.$$

Suppose that $\sigma > 0$ and that d_0 is sufficiently large. Then (VIII.13) and (VIII.4) give (for ψ)

$$\begin{aligned} g(s) &= \frac{1}{2\pi i} \sum b_n \int_{d_0 - i\infty}^{d_0 + i\infty} \Gamma(w)(2\pi\mu_n s)^{-w}\, dw \\ &= \frac{1}{2\pi i} \int_{d_0 - i\infty}^{d_0 + i\infty} G(w, s)\, dw, \end{aligned}$$

where

$$G(w, s) = \Gamma(w)(2\pi s)^{-w}\psi(w).$$

Therefore, by (VIII.14) and (VIII.8) we can write:

$$\text{(VIII.16)} \quad G(w, s) = \tfrac{1}{2}\pi^{-(w+1)/2} s^{-w}\, \Gamma\left(\frac{w+1}{2}\right) \chi(\delta - w).$$

By setting

$$\text{(VIII.17)} \quad K(s) = \frac{1}{2\pi i} \int_C G(w, s)\, dw,$$

where C is a rectifiable curve enclosing K (the integral being taken in the positive sense) we can also write for $\sigma > 0$, by applying Cauchy's theorem (d_0 being sufficiently large), with (VIII.6) taken into account

$$\begin{aligned} g(s) &= K(s) + \frac{1}{2\pi i} \int_{\delta - d_0 - i\infty}^{\delta - d_0 + i\infty} G(w, s)\, dw \\ &= K(s) + \frac{1}{2\pi i} \int_{d_0 - i\infty}^{d_0 + i\infty} G(\delta - w, s)\, dw, \end{aligned}$$

with

$$G(\delta - w, s) = \tfrac{1}{2}\pi^{-(\delta+1)/2} s^{-\delta} \pi^{w/2} s^{w}\, \Gamma\left(\frac{\delta - w + 1}{2}\right) \chi(w).$$

The formula (VIII.7) allows us to write

$$G(\delta - w, s) = \tfrac{1}{2}\pi^{-(\delta+1)/2} s^{-\delta+w}\, \Gamma\left(\frac{w}{2}\right) \Gamma\left(\frac{\delta - w + 1}{2}\right) \varphi(w).$$

By (VIII.4) (defining φ) and (VIII.15), where we put $p=q=(\delta+1)/2$, we have:

$$\frac{1}{4\pi i}\int_{d_0-i\infty}^{d_0+i\infty} s^w \Gamma\left(\frac{w}{2}\right)\Gamma\left(q-\frac{w}{2}\right)\varphi(w)\,dw$$
$$= \Gamma(q)\sum a_n\left[\left(1+\frac{\lambda_n^2}{s^2}\right)^{-q} + \sum_{0\leqslant 2j<d_0-2q} A_j\left(\frac{s^2}{\lambda_n^2}\right)^{q+j}\right].$$

This finally gives (VIII.10). As to (VIII.12), we see this immediately by (VIII.16) and (VIII.17).

If two pairs (φ_1, ψ_1), (φ_2, ψ_2) satisfy the Riemann functional equation corresponding to (Λ, M, δ) then for arbitrary constants c_1 and c_2, the pair

$$(c_1\varphi_1 + c_2\varphi_2, \quad c_1\psi_1 + c_1\psi_2)$$

satisfies the same equation. A set of k pairs

$$(\varphi_j, \psi_j) \quad (j = 1, 2, \ldots, k)$$

satisfying (VIII.5) is said to be *linearly dependent* if there exist constants c_j $(j=1, 2, \ldots, k)$, not all zero, such that

$$c_1\varphi_1 + c_2\varphi_2 + \cdots + c_k\varphi_k \equiv 0,$$
$$c_1\psi_1 + c_2\psi_2 + \cdots + c_k\psi_k \equiv 0.$$

If this is not the case, we say that the k pairs are *linearly independent*. The following theorem gives an indication of the number of linearly independent solutions which a Riemann functional equation corresponding to (Λ, M, δ) (where the sequences Λ, M and the constant $\delta>0$ are given) can have:

Theorem VIII.1.2. *Let $\bar{D}^{\cdot}_{\mu}$ be the mean upper density of M and assume that $\bar{D}^{\cdot}_{\mu}<\infty$.*

The number of linearly independent solutions of the Riemann functional equation corresponding to (Λ, M, δ) is equal to the smallest number of λ_n $(\lambda_n \in \Lambda)$ lying in an interval on $[0, \infty[$ of length exceeding $\bar{D}^{\cdot}_{\mu}$.

Let $[a, b]$ be an interval $a>0$, $b-a>\bar{D}^{\cdot}_{\mu}$, containing the points $\lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_{p+k}$ and not containing other λ_n. We are going to show that there exist at most k independent solutions of (VIII.5) corresponding to (Λ, M, δ).

Let (φ_q, ψ_q) $(q=1, 2, \ldots, k)$ be k solutions of (VIII.5). If these solutions were linearly dependent no matter how they are chosen, the theorem would be proved. Suppose then that this set of solutions can be chosen linearly independent.

We set

$$\varphi_q(s) = \sum \frac{a_n^{(q)}}{\lambda_n^s}, \qquad \psi_q(s) = \sum \frac{b_n^{(q)}}{\mu_n^s}.$$

The quantities $a_{p+1}^{(q)}, a_{p+2}^{(q)}, \ldots, a_{p+k}^{(q)}$, therefore are such that for no set of k constants $c_1, c_2, \ldots, c_k$, not all zero, we can have

$$\text{(VIII.18)} \quad \sum_{1 \leqslant q \leqslant k} c_q a_{p+1}^{(q)} = 0 \quad (j = 1, 2, \ldots, k).$$

For if (VIII.18) were to hold (for a choice of the constants $c_1, c_2, \ldots, c_k$, not all zero), then the function

$$F(s) = \sum \Bigl[\sum_{1 \leqslant q \leqslant k} c_q b_n^{(q)} \, e^{-2\pi\mu_n s} \Bigr] = \sum d_n \, e^{-2\pi\mu_n s}$$

would have the following properties:

(1) $F(s)$ would be holomorphic, by VIII.1.1, in every strip $a < t < b$,

(2) in this strip, again by VIII.1.1, we would have $F(s) = O(|\sigma|^r)$ as σ tends to $-\infty$ (r being constant). Denoting by μ a constant $0 < \mu < \mu_1$ and setting $F_1(s) = F(s)\, e^{2\pi\mu s}$, we would have

$$F_1(s) = \sum d_n \, e^{-2\pi(\mu_n - \mu)s} = \sum d_n \, e^{-\nu_n s},$$

where $F_1(s)$ is holomorphic and bounded in the strip $a < t < b$. But the upper mean density $\bar{D}_\nu^{\boldsymbol{\cdot}}$ of $\{\nu_n\}$ is equal to $\bar{D}_\mu^{\boldsymbol{\cdot}}/2\pi$. Thus, $F_1(s)$, whose abscissa of absolute convergence is finite $(=0)$, would be holomorphic and bounded in a horizontal strip of width exceeding $2\pi \bar{D}_\nu^{\boldsymbol{\cdot}}$. By Theorem III.2.3, $F_1(s)$ would be identically zero, contrary to our assumption that the pairs (φ_q, ψ_q) are linearly independent.

We have just shown that the linear independence of the solution (φ_q, ψ_q) $(q=1, 2, \ldots, k)$ implies that of the k systems $(a_{p+1}^{(q)}, a_{p+2}^{(q)}, \ldots, a_{p+k}^{(q)})$. For any $(k+1)$-th system $(a_{p+1}^{(k+1)}, a_{p+1}^{(k+1)}, \ldots, a_{p+k}^{(k+1)})$ there exists constants $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that

$$a_{p+j}^{(k+1)} = \alpha_1 a_{p+j}^{(1)} + \alpha_2 a_{p+j}^{(2)} + \cdots + \alpha_k a_{p+j}^{(k)} \quad (j = 1, 2, \ldots, k).$$

Thus, if we consider a solution $(\varphi_{k+1}, \psi_{k+1})$ of (VIII.5) written as

$$\varphi_{k+1}(s) = \sum \frac{a_n^{(k+1)}}{\lambda_n^s}, \qquad \psi_{k+1}(s) = \sum \frac{b_n^{(k+1)}}{\mu_n^s},$$

we can state, by applying VIII.1.1 once more, that for a certain constant μ' with $0<\mu'<\mu_1$ the function

$$\begin{aligned} F_2(s) &= \sum d_n'\, e^{-2\pi(\mu_n-\mu')s} \\ &= \sum \Big[\sum_{1\leqslant q\leqslant k} \alpha_q b_n^{(q)}\, e^{-2\pi(\mu_n-\mu')s} \Big] - \sum b_n^{(k+1)}\, e^{-2\pi(\mu_n-\mu')s} \end{aligned}$$

is holomorphic and bounded in the strip $a<t<b$ (whose width exceeds $2\pi\bar{D}^{\bullet}_{\nu'}$, where $\bar{D}^{\bullet}_{\nu'}$ is the mean upper density of $\{\nu_n'\}$ with $\nu_n'=2\pi(\mu_n-\mu')$). From Theorem III.2.3 we deduce again that $F_2(s)\equiv 0$; that is to say,

$$(\varphi_{k+1}, \psi_{k+1}) = (\alpha_1\varphi_1 + \cdots + \alpha_k\varphi_k, \alpha_1\psi_1 + \cdots + \alpha_k\psi_k).$$

This proves the theorem.

The next theorem gives conditions which the sequences Λ, M and the constant δ must satisfy so that the Riemann functional equation corresponding to (Λ, M, δ) should have a solution. This theorem shows essentially that if the step of M is positive ($h_\mu>0$), then Λ has an 'integral basis' which incidentally can be taken in an arbitrary interval of length exceeding $\bar{D}^{\bullet}_{\mu}+h_\mu^{-1}$.

THEOREM VIII.1.3. *Let M be a sequence of positive step h_μ. Let $\bar{D}^{\bullet}_{\mu}$ be the upper mean density of M. Let $\delta=1$ or $\delta=3$ and suppose that the sequences Λ and A are such that there exist three constants $\mathfrak{H}$, $\mathfrak{C}$, $\mathfrak{N}\geqslant 1$ such that for $n\geqslant 1$*

(VIII.19) $\mu_{n+1}-\mu_n < \mathfrak{H}$

(VIII.20) $\mathfrak{C}^{-1} \leqslant \max\limits_{1\leqslant\nu\leqslant\mathfrak{N}} |b_{n+\nu}| \leqslant \mathfrak{C}$, when $\delta = 1$;

$\mathfrak{C}^{-1} \leqslant \max\limits_{1\leqslant\nu\leqslant\mathfrak{N}} |b_{n+\nu}\mu_{n+\nu}^{-1}| \leqslant \mathfrak{C}$, when $\delta = 3$.

Suppose that the pair

$$\varphi(s) = \sum \frac{a_n}{\lambda_n^s}, \quad \psi(s) = \sum \frac{b_n}{\mu_n^s},$$

satisfies the Riemann equation corresponding to (Λ, M, δ).

Let p and k be two positive integers such that

$$\lambda_{p+k+1} - \lambda_p > \bar{D}^{\bullet}_{\mu} + h_\mu^{-1}.$$

Then $k\geqslant 2$, and every λ_n is of the form

(VIII.21) $\lambda_n = m_1^{(n)}\lambda_{p+1} + m_2^{(n)}\lambda_{p+2} + \cdots + m_k^{(n)}\lambda_{p+k}$,

where the m_j ($j=1, 2, \ldots, k$) are integers.

There exists at least one λ_n *exceeding* $\frac{1}{2}(\bar{D}^{\cdot}_{\mu}+h_{\mu}^{-1})$. *If* q *is the smallest integer such that* $\lambda_{q+1}>\frac{1}{2}(\bar{D}^{\cdot}_{\mu}+h_{\mu}^{-1})$, *then every* λ_n *is of the form*

$$\text{(VIII.22)} \quad \lambda_n = p_1^{(n)}\lambda_1 + p_2^{(n)}\lambda_2 + \cdots + p_q^{(n)}\lambda_q,$$

where the p_j $(j=1, 2, \ldots, q)$ *are integers.*

Furthermore,

$$\mu_{n+1} - \mu_n \geqslant h_\mu \quad (n \geqslant 1).$$

If we set for $\sigma>0$*:*

$$\Psi(s) = (B, 2\pi M) = \sum b_n\, e^{-2\pi\mu_n s}, \quad \text{when} \quad \delta = 1$$
$$\Psi(s) = (BM^{-1}, 2\pi M) = \sum b_n\mu_n^{-1}\, e^{-2\pi\mu_n s}, \quad \text{when} \quad \delta = 3,$$

then the function $\Psi(s)$ *is holomorphic and uniform outside the axis of convergence* $\sigma=0$, *where its only singularities are simple poles at the places* $\pm i\lambda_n$ *and* $s=0$. *For* $\sigma<0$ *and suitable constants c and d we have:*

$$\Psi(s) = -\sum b_n\, e^{2\pi\mu_n s} + c, \quad \text{when} \quad \delta = 1$$
$$\Psi(s) = \sum b_n\mu_n^{-1}\, e^{2\pi\mu_n s} + d, \quad \text{when} \quad \delta = 3.$$

We remark that the conditions (VIII.19) and (VIII.20) in the statement of Theorem VIII.1.3 are superfluous: the conclusion is valid without these assumptions; and these conditions themselves are consequences of the other assumptions of the theorem. The present theorem, incidentally, is an almost immediate consequence of Theorem VII.2.1, where, as we have remarked, the conditions (VII.20) and (VII.21) follow from the other assumptions of the same theorem, namely from the fact that the series (A, Λ) only has simple poles on an interval of a certain length on the axis of convergence. Here the conditions (VIII.19) and (VIII.20) follow from the fact that (φ, ψ) is a solution of the Riemann functional equation.

Here is a proof of Theorem VIII.1.3 on a few lines. The fact that $k\geqslant 2$ is an immediate consequence of Theorem III.2.3. For if we had $0\leqslant k\leqslant 1$, then there would exist a sufficiently small $\varepsilon>0$ so that

$$I = (\lambda_{p+2} - \varepsilon) - (\lambda_p + \varepsilon) > \bar{D}^{\cdot}_{\mu} + h_{\mu}^{-1}.$$

And an interval of length I would contain at most a single λ_n (namely λ_{p+1}). And since $\bar{D}^{\cdot}_{\mu}\leqslant h_{\mu}^{-1}$, there would exist an interval I_1 of length exceeding $\bar{D}^{\cdot}_{\mu}$ not containing any λ_n.

The function $\Psi(s)$, which has on the axis of the convergence $\sigma=0$ only simple poles at the places $\pm i\lambda_n$ and (by Theorems VII.2.1 and VIII.1.1) the point $s=0$, would be holomorphic in the horizontal strip that cuts out on $\sigma=0$ the interval I_1; in this strip the function would be $O(|\sigma|^m)$ $(\sigma \to -\infty)$. Theorem III.2.1 would then show that $\psi\equiv 0$. We could also apply Theorem VIII.1.2 directly, which itself is an almost immediate consequence of Theorems III.2.3 and VIII.1.1.

Now (VIII.21) follows immediately from VII.2.1. And (VIII.22) follows from the fact that on an interval $[-i(\lambda_{q+1}-\varepsilon), i(\lambda_{q+1}-\varepsilon)]$ (where $\varepsilon>0$ is sufficiently small so that the length of this interval exceeds $\bar{D}^{\cdot}_{\mu}+h_{\mu}^{-1}$), $\Psi(s)$ has as singularities only the simple poles at the places $\pm i\lambda_n$ $(1\leqslant n\leqslant q)$ and the point $s=0$ (VIII.22) therefore does not differ from (VIII.21). The other conclusions of the theorem are reached by applying first VII.2.1 to $\Psi(s)\,e^{\mu s}$ $(0<\mu<2\pi\mu_i)$. Therefore, by VIII.1.1, we see that for $\delta=1$ or $\delta=3$, the function $\Psi(s)$ (minus a constant) is odd or even, respectively.

Here are some corollaries to Theorem VIII.1.3. We assume tacitly that the conditions (VIII.19) and (VIII.20) are satisfied. We also assume that the pair (φ, ψ) satisfies the Riemann functional equation corresponding to (Λ, M, δ) with $\delta=1$ or $\delta=3$.

VIII.1.4. *If $h_\mu\lambda_2>1$, then there exists an increasing sequence of positive integers $\{m_n\}$ such that $\lambda_n=m_n\lambda_1$ $(n\geqslant 1)$; there exists an integer ν such that $\mu_\nu\lambda_1=1$, $b_n=b_q$ for $n\equiv q$ (mod ν).*

By Theorem VIII.1.3 we have: $0<\lambda_1<\frac{1}{2}(\bar{D}^{\cdot}_{\mu}+h_{\mu}^{-1})$; and since, by hypothesis, $\lambda_2>h_{\mu}^{-1}\geqslant\frac{1}{2}(\bar{D}^{\cdot}_{\mu}+h_{\mu}^{-1})$, the same Theorem VIII.1.3 gives $\lambda_n=m_n\lambda_1$ $(n\geqslant 1)$.

Thus, the function

$$\Phi(s) = \sum a_n\, e^{-2\pi\lambda_n s}$$

is periodic of period $i\lambda_1^{-1}$. Since the only singularities of Φ are the points $\pm i\mu_n$ and the origin (by VIII.1.1), there exists an integer ν such that $\mu_\nu=\lambda_1^{-1}$. Consequently, for every n we have: $\mu_n=\mu_q+m\lambda_1^{-1}$, where $0\leqslant q<\nu$ $(\mu_0=0)$, and m is an integer. From the periodicity of Φ it also follows that $b_n\equiv b_q$(mod ν).

Remark. Since the points $\pm i\mu_n$ and the origin are the only singularities of $\Phi(s)$, namely simple poles of Φ (if $\delta=1$) or of the function $\sum a_n\lambda_n^{-1}\,e^{-2\pi\lambda_n s}$ (if $\delta=3$) we conclude easily, owing to the periodicity of Φ and of that of the places and the residues of the poles, that the a_n and the λ_n satisfy the conditions (VIII.19) and (VIII.20), with the b_n and the μ_n replaced, respectively, by the a_n and the λ_n (for a suitable choice of the constants $\mathfrak{H}$, $\mathfrak{C}$, $\mathfrak{N}$).

VIII.1.5. *If* $h_\mu \lambda_2 > 1$ *and if* $2h_\mu \lambda_1 > 1$, *then* $\lambda_n = n\lambda_1$ $(n \geqslant 1)$.

By VIII.1.4 we have: $\lambda_n = m_n \lambda_1$, where m_n is an increasing sequence of positive integers. If the conclusion of VIII.1.5 did not hold, then there would exist an n_0 such that

$$\lambda_{n_0+1} - \lambda_{n_0} \geqslant 2\lambda_1,$$

and the condition $2h_\mu \lambda_1 > 1$ would give

$$\lambda_{n_0+1} - \lambda_{n_0} > h_\mu^{-1} \geqslant \bar{D}^{\cdot}_\mu;$$

and III.2.3 would again lead to the conclusion $\Phi \equiv 0$.

Remark. The conditions of VIII.1.5 are satisfied if $h_\mu \lambda_1 \geqslant 1$.

VIII.1.6. *If to the assumptions of* VIII.1.5 *we add* $\mu_2 \lambda_1 > 1$, *then there exists an increasing sequence of positive integers* $\{m'_n\}$ *such that* $\mu'_n = m'_n \mu_1$ *and* $\lambda_1 \mu_1$ *is a rational number.*

We refer to the remark after the proof of VIII.1.4. By VIII.1.5, we have $\lambda_n = n\lambda_1$, hence h_λ, the step of λ, is equal to λ_1; the equality $\mu_n = m'_n \mu_1$ $(n \geqslant 1)$ therefore follows from VIII.1.4. And since $\mu_n = \mu_q + m\lambda_1^{-1}$ $(0 \leqslant \mu_q \leqslant \lambda_1^{-1}, \mu_0 = 0)$ (see the proof of VIII.1.4), we conclude that for an n with $\mu_n > \lambda_1^{-1}$ we have $m'_n \mu_1 = m'_q \mu_1 + m\lambda_1^{-1}$, with $m'_n - m'_q > 0$, that is, $\lambda_1 \mu_1 = m/(m'_n - m'_q)$.

VIII.1.7. *If* $h_\mu \lambda_2 > 1$, $2h_\mu \lambda_1 > 1$, $\mu_2 \lambda_1 > 1$, $2\mu_1 \lambda_1 > 1$, *then* $\mu_n = n\mu_1$ $(n \geqslant 1)$.

This follows immediately from VIII.1.5 and the first part of VIII.1.6.

In particular, if $\lambda_1 = 1$, $\mu_n = n$, then also $\lambda_n = n$.

VIII.1.8. *If* $\mu_n = n$ $(n \geqslant 1)$ *and if* $\lambda_1 \geqslant 1$, *then every solution* (φ, ψ) *of* (Λ, M, δ) *is equal to* $a\zeta(s)$, *where* a *is a constant and* ζ *is the Riemann zeta-function.*[1]

This follows easily from VIII.1.5 and the argument used in the proof of VIII.1.4 (which here yields $a_n = a_{n+1} = b_1$ $(n \geqslant 1)$).

NOTE

[1] We recall that from Theorem VIII.1.3 onwards we assume that $\delta = 1$ or $\delta = 3$.

CHAPTER IX

INFLUENCE OF ARITHMETICAL PROPERTIES OF THE EXPONENTS OF A DIRICHLET SERIES ON ITS ANALYTIC CONTINUATION

IX.1. Isolated fractional parts of the exponents and the possibility of analytic continuation

There is an interesting link between two apparently distinct phenomena in the theory of functions: on the one hand, the dependence of the character of a function represented by a Dirichlet series (and above all, of the distribution of its singularities) on the arithmetic structure of its exponents, and on the other hand, the behaviour of a function of class A (that is, of a Fourier transform of a function belonging to *L*) with compact support, or what comes to the same thing, by the Paley-Wiener theorem, the behaviour of an entire function of exponential type that is integrable over the real axis.

A very special case of this link between the two phenomena has already occurred in Chapter V in the proof of Theorem V.4.3, in which Lemma V.4.1 played an important role. But, in fact, the lemma in question itself gives a straightforward link between the behaviour of an entire function of exponential type on the parallels to the real axis, and the type and the order of magnitude of the maximum of the Fourier transform of the function will only be fully applied in this chapter, in (IX.2) more completely than in (IX.1). Here we are no longer concerned with the question of 'composition of singularities', which was treated in Chapter V; we shall speak of the 'distribution of singularities', although some important points of the technique we use are akin to that of Chapter V.

In what follows we assume, as before, that the step of the sequence Λ is positive. We also assume that the abscissa of convergence of (A, Λ) is $\sigma_a = 0$; we recall that when the step is positive,

$$\sigma_c = \sigma_a = \limsup (\log |a_n|/\lambda_n).$$

We suppose that the continuation (which we assume to be direct) of $f(s) = (A, \Lambda)$ into $P_{\sigma'}$ with $\sigma' < \sigma_a$ is uniform. If $S_{\sigma'}$ $(= S_{\sigma', f})$ is the singular set of f in $P_{\sigma'}$, we denote by $S^*_{\sigma'}$ $(= S^*_{\sigma', f})$ the closure of the set of all points $\alpha + 2k\pi i$, where α is an arbitrary point of $S_{\sigma'}$ and k is an arbitrary integer, then we denote by Σ the intersection of $S^*_{\sigma'}$ with the axis of convergence.

This set Σ is independent of σ' if $\sigma' < \sigma_a$. If Σ is not the line $\sigma = \sigma_a$, we say that *the series $f = (A, \Lambda)$ is continuable modulo $2\pi i$.* Otherwise we say that *the series has the axis of convergence as natural boundary modulo $2\pi i$.* If the points of Σ are isolated points of $S^*_{\sigma'}$, we say that *the singular set of f on the axis of convergence is isolated modulo $2\pi i$.*

We denote by (λ_n) the fractional part of λ_n, that is, $\lambda - E\lambda_n$ (we recall that Ea is the integral part of a).

The sequence $\{(\lambda_n)\}$ of fractional parts can also be denoted by (Λ).

The theorems which we are going to prove first of all have the following meaning: if $f = (A, \Lambda)$ can be continued modulo $2\pi i$ across its axis of convergence, and if f does not grow in $P_{\sigma'}$ $(\sigma' < \sigma_a)$, except at the singularities, faster than a function $A(t)$ (which itself is subject to certain conditions), then the values taken by the fractional parts (λ_n) must be sufficiently 'piled up', the 'piling' depending on the growth of $A(t)$. The piling is still greater (for example, almost all isolated values of the set values taken by the (λ_n) must each be taken infinitely many times) if the singular set of f on the axis of convergence is isolated modulo $2\pi i$.

To make precise statements of these theorems we need some new definitions.

We denote by Ω the set of values taken by the members of (Λ), that is, by the fractional parts (λ_n) of the λ_n. If $\omega \in \Omega$, then there exists a $\lambda_n \in \Lambda$ such that $(\lambda_n) = \omega$, and $0 \leqslant \omega < 1$, $\lambda_n - \omega$ an integer; but there may be several λ_n, even infinitely many, whose fractional part is equal to ω. We denote by Λ_ω the set of all the λ_n for which $(\lambda_n) = \omega$. Let $\lambda^{(\omega)}$ be the smallest term of Λ_ω and $\nu^{(\omega)}$ the number, possibly infinite, of elements in Λ_ω. If $\omega \neq 0$, where ω is an isolated point of Ω, we denote by $\delta^{(\omega)}$ the smallest distance from ω to the other points of the closure of Ω (if such points exist). The interval $I^{(\omega)} =]\omega - \delta^{(\omega)}, \omega + \delta^{(\omega)}[$ is called *the interval of isolation of ω.* In the interval of isolation of ω there is no other point of Ω, and one of its extremities belongs to $\bar{\Omega}$.

In this section we prove the following theorems.

THEOREM IX.1.1. (1) *Suppose that $f = (A, \Lambda)$ is continuable modulo $2\pi i$ across its axis of convergence $(\sigma = 0)$ into $P_{\sigma'}$ $(\sigma' < 0)$. Suppose that f does not grow in $P_{\sigma'}$, except at the singularities, faster than $A(t)$, where $A(t)$ $(t \geqslant 0)$ is a positive non-decreasing function satisfying the condition*

$$\text{(IX.1)} \qquad \int_1^\infty \frac{\log A(t)}{t^2}\, dt < \infty.$$

(2) *Suppose that when we set*

$$\text{(IX.2)} \quad h(a) = \int_a^\infty \frac{\log A(t)}{t^2}\,dt \quad (a > 0),$$

the set Ω of values ω taken by the terms of the sequence (Λ) of fractional parts of the λ_n contains an infinite sequence $\{\omega_n\}$ of isolated values in Ω satisfying the condition

$$\text{(IX.3)} \quad h(\lambda^{(\omega_n)}) \leqslant c\,\delta^{(\omega_n)} \quad (n \geqslant 1),$$

where c is a constant smaller $(2e)^{-1}$.

(3) *In every Λ_{ω_n} there exists a λ_{m_n} such that*

$$\text{(IX.4)} \quad \lim \frac{\log |a_{m_n}|}{\lambda_{m_n}} = 0\ (= \sigma_a).$$

The conclusion is:

$$\liminf \frac{\nu^{(\omega_n)}}{\lambda^{(\omega_n)}} > 0.$$

We recall that $\limsup \log(|a_n|/\lambda_n) = 0$ because the step of Λ is positive and $\sigma_a = 0$.

Thus, the order of magnitude of the number of times the isolated value ω_n of Ω is taken (where the value of ω_n is useful in the sense of (IX.4); that is, the ω_n are the fractional parts of the λ_{m_n} which with the corresponding coefficients $|a_{m_n}|$ provide by means of (IX.4) a limit equal to the superior limit taken over all the λ_n and $|a_n|$, which yields the abscissa of convergence; here $\sigma_a = 0$) by the terms of the sequence (Λ) is at least equal to that of the smallest term of Λ whose fractional part is equal to ω_n.

If $A(t) \equiv 1$, then $h(a) \equiv 0$, and the condition (IX.3) is automatically satisfied. Hence Theorem IX.1.1 leads to the following theorem:

THEOREM IX.1.2. *The conclusion of Theorem* IX.1.1 *remains valid if the conditions in this theorem are replaced by the following condition:*

The series $f = (A, \Lambda)$ is continuable modulo $2\pi i$ across the axis of convergence into the half-plane $P_{\sigma'}$ $(\sigma' < 0)$, where the continuation of f is bounded, except at the singularities. The set of values taken by (Λ) contains an infinite sequence $\{\omega_n\}$ of isolated values in Ω satisfying condition (3) *of Theorem* IX.1.1.

We are going to construct an example of a series (A, Λ) that satisfies all the conditions of Theorem IX.1.2 (which, as we have just seen, is nothing but a particular case of Theorem IX.1.1), but is not a Taylor-D series.

Consider, on the one hand, a sequence of positive even integers $M=\{\mu_n\}$ such that $\mu_{n+1}>2\mu_n$, and on the other hand, a positive sequence $\{\omega_n\}$, with $\omega_1<1$, decreasing to zero; in addition, the sequences are such that for a certain value $a<-\log 2$

$$\text{(IX.6)} \qquad \sum \omega_n\, e^{-2\mu_n a} < \infty,$$

and let us study the function

$$\text{(IX.7)} \qquad f(s) = \sum e^{-\omega_n s}\left(\frac{e^{-s}+e^{-2s}}{2}\right)^{\mu_n}.$$

Two polynomials $(z+z^2)^{\mu_n}$ (corresponding to two different values of n) have no power in common owing to the fact that $\mu_{n+1}>2\mu_n$; we also see that $f(s)$ for $\sigma>0$ can be put in the form

$$f(s) = \sum a_m\, e^{-\lambda_m s} = (A, \Lambda),$$

where the λ_m range over all the numbers of the form ω_n+q with $\mu_n \leqslant q \leqslant 2\mu_n$ $(n\geqslant 1)$. From the inequalities $1>\omega_1>\omega_2>\cdots$; $\mu_{n+1}>2\mu_n+1$ it follows that Λ has step $h>0$, because $h\geqslant 1-\omega_1$. Since $(\lambda_m)=\omega_n$ for $\lambda_m=\omega_n+q$, we see that $\Omega\equiv\{\omega_n\}$ and it is clear that $\lambda^{(\omega_n)}=\mu_n+\omega_n$, $\nu^{(\omega_n)}=\mu_n+1$. We also see that when $\lambda_m=\omega_n+q$, then

$$a_m = 2^{-\mu_n}\binom{\mu_n}{q-\mu_n}.$$

By setting $\lambda_{m_n}=\omega_n+3\mu_n/2$ we see that condition (3) of Theorem IX.1.1 is satisfied. We also see that $\sigma_a=0$.

On the other hand, it is easy to see that f is holomorphic in the domain D bounded by the curve C, given by $|e^{-s}+e^{-2s}|=2$ and containing the half-line $\sigma>0$, and that this curve is a natural boundary for f.

In fact, the function

$$f_1(s) = \sum\left(\frac{e^{-s}+e^{-2s}}{2}\right)^{\mu_n}$$

certainly has these properties, by virtue of Hadamard's 'lacunary' theorem, which is a very special case of Theorem VI.2.2 (the series $\sum z^{\mu_n}$ has the circle of convergence $|z|=1$ as natural boundary); and for $\sigma>0$

we have:

$$\text{(IX.8)}\quad f(s) - f_1(s) = \sum (e^{-\omega_n s} - 1)\left(\frac{e^{-s} + e^{-2s}}{2}\right)^{\mu_n}$$

so that on every compact set contained in P_a, with the chosen value of a (where $a < -\log 2$ and (IX.6) holds):

$$\text{(IX.9)}\quad |f(s) - f_1(s)| \leqslant K \sum \omega_n |s|\, e^{-2\mu_n a} \leqslant K_1 \sum \omega_n\, e^{-2\mu_n a} < \infty,$$

K and K_1 being constants.

Since $\bar{D} \subset P_a$ (because $a < -\log 2$), we see that $f(s)$ is holomorphic in D; it follows from (IX.8) and (IX.9) that $f(s) - f_1(s)$ is holomorphic in P_a; since the boundary C of D, as we have seen above, is a natural boundary for f_1, it is also one for f.

The inequality

$$|f(s) - f_1(s)| \leqslant \sum (1 + e^{-\omega_n \sigma})\delta^n,$$

with $0 < \delta < 1$ ($\delta = \delta(\varepsilon)$), which by (IX.8) is valid on every part $\bar{D}_\varepsilon$ of D situated at a distance not less than ε ($\varepsilon > 0$) from C, shows that $f(s)$ is bounded on $\bar{D}\varepsilon$ (because $f_1(s)$ is bounded there). Therefore $f(s)$ satisfies all the conditions of Theorem IX.1.2.

Here evidently

$$\lim \frac{\nu^{(\omega_n)}}{\lambda^{(\omega_n)}} = 1.$$

Hence there exists a Dirichlet series that is not a Taylor-D series but satisfies all the conditions of Theorem IX.1.1 and even those of the special case Theorem IX.1.2. But these theorems indicate that a Dirichlet series satisfying their conditions has one important point in common with a Taylor-D series: if there exist isolated fractional values of the exponents corresponding to 'useful' λ_n (in the sense of condition (3) of Theorem IX.1.1), then the number of terms of Λ that have each of these values as fractional part is at least of the same order of magnitude as the smallest of these terms.

We now proceed to the proof of Theorem IX.1.1. Suppose that the theorem is not true. Under these conditions we can extract from $\{\omega_n\}$ a partial sequence, which we denote by $\{\alpha_n\}$, having the following properties.

(1) α_n tends monotonically to a quantity α^0; we may assume that α_n

decreases to α^0, because an insignificant change in the subsequent argument is sufficient to treat the case when α_n increases to α^0.

(2) $0<\alpha_n-\alpha^0<\frac{1}{2}\inf(\lambda_{n+1}-\lambda_n)=r; \qquad \alpha_1+\eta^{(\alpha_1)}<1.$

(3) $\nu^{(\alpha_n)}=o(\lambda^{(\alpha_n)})$.

(4) $\sum\limits_{k\leqslant n-1}\nu^{(\alpha_k)}=o(\lambda^{(\alpha_n)})$.

(5) $\lambda^{(\alpha_{n+1})}$ is larger than the largest term of Λ_{α_n}. (We recall that Λ_ω denotes the set of λ_n whose fractional part is equal to ω).

(6) $\alpha_n-\delta^{(\alpha_n)}>\alpha_{n+1}+\delta^{(\alpha_{n+1})} \quad (n\geqslant 1)$.

Denote by Λ' the sequence $\{\lambda_n'\}$, arranged in order of magnitude, that is formed by the term of $\bigcup \Lambda_{\alpha_\nu}$. Then Λ' is composed of all the λ_m belonging to all the Λ_{α_ν} $(\nu\geqslant 1)$.

Let $m_n=E\lambda_n'$. Then $\lambda_n'=m_n+\varepsilon_n'$, with $0<\varepsilon_n'-\alpha^0<r$, ε_n' decreasing to α^0. To every n there corresponds a ν such that $\varepsilon_n'=\alpha_\nu$; α_ν decreases to α^0. We see that m_n increases to infinity, and from (3) and (4) it follows that

$$\text{(IX.10)} \quad \lim \frac{n}{m_n}=0.$$

Denote by λ_{r_ν} a term of Λ_{α_ν} chosen so that

$$\text{(IX.11)} \quad \lim \frac{\log |a_{r_\nu}|}{\lambda_{r_\nu}}=0,$$

which is possible by virtue of (IX.4).

Now we construct the functions $F_a(z)$ of V.4.1 corresponding to the elements specified there, in the following manner: if $x_0>0$ is such that $A(x_0)\geqslant 1$, then we set $C(x)=0$ for $0<x<x_0$, and $C(x)=\log A(x)$ for $x\geqslant x_0$. For the quantity p occurring in V.4.1 we take $(ce)^{-1}$ and for q (in the same theorem) an arbitrary quantity such that $0\leqslant q<(ce)^{-1}-2$. When we set $a=\lambda^{(\alpha_n)}$, the corresponding function (of V.4.1) is denoted by $\Phi_n(z)$.

Since for every $a>0$:

$$\text{(IX.12)} \quad h(a)=\int_a^\infty \frac{C(t)}{t^2}\,dt \geqslant \frac{C(a)}{a},$$

it follows from the construction of F_a (see (V.16), (V.17) and (V.18)) that $\Phi_n(z)$ is an even entire function satisfying the following conditions:

$$\text{(IX.13)} \quad |\Phi_n(z)| \leqslant e^{\delta^{(\alpha_n)}|y| - C(|z|)} \mathscr{L}_n(x),$$

where $\mathfrak{L}_n$ is an even positive function decreasing for $x \geqslant 0$, $\mathscr{L}_n \in L$, $\|\mathscr{L}_n\| = 1$; the Fourier transform $\hat{\Phi}_n(u)$ of $\Phi_n(x)$ is real, non-negative, even, and satisfies the conditions

$$\text{(IX.14)} \quad \hat{\Phi}_n(u) \geqslant g\, e^{-\varepsilon_n \lambda^{(\alpha_n)}} \quad \text{for} \quad |u| \leqslant l\delta^{(\alpha_n)},$$

$$\text{(IX.15)} \quad \hat{\Phi}_n(u) = 0 \quad \text{for} \quad |u| \geqslant \delta^{(\alpha_n)},$$

where $\varepsilon_n = h(\lambda^{(\alpha_n)})$ is a quantity tending to zero, g is a numerical constant, and $0 \leqslant l < (1 - 2ce)$.

We set

$$\text{(IX.16)} \quad \Phi(z) = \sum \frac{1}{n^2} e^{i\alpha_n z} \Phi_n(z).$$

Since $\Phi(z)$ is an entire function, we have by (IX.13) and the definition of $C(x)$,

$$\text{(IX.17)} \quad |\Phi(z)| \leqslant [A(|z|)]^{-1} e^{d|y|} L(x),$$

where d is a constant and $L(x)$ decreases to zero as $|x|$ tends to infinity; $L(x) \in L$.

Denoting by $\hat{\Phi}(u)$ the Fourier transform of $\Phi(x)$ we also have:

$$\hat{\Phi}(u) = \sum \frac{1}{n^2} \hat{\Phi}_n(u - \alpha_n).$$

Taking (IX.14) and (IX.15) into account, we see that when m is a positive integer,

$$\hat{\Phi}(\lambda_n - m) = 0$$

except when λ_n is a term λ'_q of Λ', and m is the integral part of λ'_q. If the fractional part of λ'_q is α_n, then

$$\hat{\Phi}(\lambda'_q - m) = \hat{\Phi}(\alpha_n) = \frac{1}{n^2} \hat{\Phi}_n(0) \geqslant \frac{1}{n^2} e^{-\varepsilon_n \lambda^{(\alpha_n)}}.$$

We set

$$\psi(\zeta) = \frac{1}{e^\zeta - 1}$$

and consider for $\sigma > c > 0$ the integral

$$\text{(IX.18)}\quad \Psi(s) = \int_{c-i\infty}^{c+i\infty} f(z)\psi(s-z)\Phi(-iz)\,dz.$$

The conditions of the theorem and the inequality (IX.17) show that this integral has a meaning for $\sigma > c$.

For $\sigma \geqslant c' > c$ we can write (IX.18) in the following form:

$$\Psi(s) = \sum e^{-ns} \sum a_m \int_{c-i\infty}^{c+i\infty} e^{(n-\lambda_m)z}\Phi(-iz)\,dz;$$

and with the given properties of Φ, by applying Cauchy's theorem we have

$$\int_{c-i\infty}^{c+i\infty} e^{(n-\lambda_m)z}\,\Phi(-iz)\,dz = i\int e^{-i(\lambda_m-n)y}\Phi(y)\,dy = i\hat{\Phi}(\lambda_m - n).$$

Thus, for $\sigma \geqslant c'$:

$$\Psi(s) = i\sum e^{-ns}\sum a_m \hat{\Phi}(\lambda_m - n).$$

According to the remarks on $\hat{\Phi}$ made earlier we see now that:

$$\text{(IX.19)}\quad \Psi(s) = i\sum a_{n_q}\, e^{-m_q s}\frac{1}{\nu_q^2}\,\hat{\Phi}_{\nu_q}(0),$$

where the quantities m_q, n_q, ν_q are defined as follows: the integers m_q are the integral parts of the λ'_q, the λ'_q being arranged in order of magnitude and forming the set Λ', which is the union of all the sets Λ_{α_ν} $(\nu \geqslant 1)$; by virtue of (2), to every m_q there corresponds a unique integer $\nu = \nu_q$ such that $\lambda'_q \in \Lambda_{\alpha_{\nu_q}}$, that is, such that

$$\lambda'_q = m_q + \alpha_{\nu_q};$$

n_q denotes the integer for which $\lambda'_q = \lambda_{n_q}$.

Thus, we can write $\Psi(s)$ in the form:

$$\text{(IX.20)}\quad \Psi(s) = \sum c_q\, e^{-m_q s}.$$

Since $|\hat{\Phi}_n(u)|$ is bounded, we see that $\sigma_{a,\Psi} \leqslant 0$; we wish to show that $\sigma_{a,\Psi} = 0$.

There exists an ascending sequence of positive integers $\{q_n\}$ such that $w_n = \nu_{q_n}$ grows and that with r_ν defined by (IX.11) and $r_{w_n} = s_n$,

$$\lambda_{s_n} \in \Lambda_{\alpha_\nu} \quad \text{with} \quad \nu = w_n,$$

$$\lim \frac{\log |a_{s_n}|}{\lambda_{s_n}} = 0.$$

Since the step of Λ is positive and $\{\lambda_{s_n}\}$ is a subsequence of Λ, and since $\{w_n\}$ and $\{\lambda_{s_n}\}$ are increasing sequences (by (5)), there exists a constant A such that

$$1 \leqslant w_n \leqslant A\lambda^{(\alpha_{w_n})} \leqslant A\lambda_{s_n}.$$

Consequently,

$$\lim \frac{\log w_n}{\lambda_{s_n}} = 0.$$

It also follows from (IX.14) that

$$\hat{\Phi}_{w_n}(0) \geqslant g\, e^{-\eta_n \lambda_{s_n}},$$

with $\lim \eta_n = 0$ $(\eta_n > 0)$, because $\lambda_{s_n} \geqslant \lambda^{(\alpha_{w_n})}$.

Consequently we have:

$$\limsup \frac{\log |\hat{\Phi}_{w_n}(0)|}{\lambda_{s_n}} \geqslant 0.$$

Hence, $\sigma_{a,\,\Psi} \geqslant 0$ and therefore $\sigma_{a,\,\Psi} = 0$.

The radius of convergence of the series

$$\text{(IX.21)} \quad T(z) = \sum c_q z^{m_q}$$

is, consequently, unity; and since (IX.10) holds, we see, by Fabry's theorem (a special case of Theorem VI.2.2), that this series has its circle of convergence $(\sigma = 0)$ as natural boundary.

Now since $\psi(s)$ has as singularities only the points $2k\pi i$, where k is an arbitrary integer, a simple repetition of the arguments used in proving Theorem V.4.3 (a suitable division of part of the plane, use of Cauchy's theorem, etc.) shows that Ψ given by (IX.18) has in $P_{\sigma'}$ no singularities other than points of $S_{\sigma',f}$, that is, points of the form $\alpha + 2k\pi i$, where α is any point of $S^*_{\sigma',f}$ and k is an arbitrary integer. Now by the conditions of the theorem the intersection Σ of $S^*_{\sigma',f}$ with $\sigma = 0$ is not the entire axis $(\sigma = 0)$. This means that the axis is not a natural boundary of Ψ. This contradicts what we have found above under the assumption that the conclusion of the theorem is false. Here we have chosen the α_n decreasing to α^0. We would find the same contradiction assuming that the α_n increase to α^0: it is sufficient to replace in (IX.18) $\Phi(-iz)$ by $\Phi(iz)\, e^z$. Thus, our theorem is now completely proved.

A set S is said to be *an isolated part modulo* $2\pi i$ of S_d if it has the following properties: $S \subset S_d^*$ and there exists a closed Jordan curve contained in $P_d - S_d^*$ and containing S in its interior.

Clearly, in the example we have constructed earlier (after Theorem IX.1.2) no point of S_d $(=S_{d,f})$ on $\sigma=0$ $(=\sigma_a)$ can lie in an isolated part modulo $2\pi i$ of S_d (in fact, this is true, in our example, for every point of S_d). This remark shows that our example also illustrates the conditions of the following theorems:

THEOREM IX.1.3. *If to the conditions of Theorem* IX.1.1 *we add the following condition:*

(d) *for infinitely many values of n*

$$\nu^{(\omega_n)} < \infty,$$

then there exists at least one point of S_d, $d<0$, *on* $\sigma=\sigma_a$ *that does not belong to any isolated part modulo* $2\pi i$ *of* S_d.

THEOREM IX.1.4. *If to the conditions of Theorem* IX.1.2 *we add the condition* (d) *of Theorem* IX.1.3, *then the conclusion of Theorem* IX.1.3 *is still valid.*

Suppose that Theorem IX.1.3 does not hold. Then every point of S_d on $\sigma=0$ belongs to an isolated part modulo $2\pi i$ of S_d. This means that every point of $S_d^*=0$ is a member of a set that is part of S_d^* such that there exists a closed Jordan curve containing the set and contained in P_d, with no point of S_d^* on the curve.

From the sequence $\{\omega_n\}$ we can now extract a sequence $\{\alpha_n\}$ which by virtue of property (d) in Theorem IX.1.3 has the properties (1) and (6) on p. 143 and 144,[1] and if $\lambda_1^{(\alpha_n)}$ denotes the largest term of Λ_{α_n}, also the property

$$\lim \frac{\lambda^{(\alpha_{n+1})}}{\lambda_1^{(\alpha_n)}} = \infty.$$

Keeping to the notation and to all the operations in the proof of Theorem IX.1.1, it suffices to make the following remarks: according to what we have just said about the points of S_d^* on $\sigma=0$ $(=\sigma_c)$ (if the theorem is supposed to be false), the function $T(z)$ given by (IX.21) has on its circle of convergence $|z|=1$ only singular points each of which can be enclosed by a Jordan curve inside the disk $|z|<e^{-d}$ and on which $T(z)$ is holomorphic. But this time the sequence $\{m_n\}$ has a property which can be expressed as follows: there exists a subsequence $\{m_{n_p}\}$ of $\{m_n\}$ with

$$\lim \frac{m_{n_p+1}}{m_{n_p}} = \infty.$$

Now by a theorem of Ostrowski, the domain of existence of a function whose Taylor expansion has such gaps is necessarily simply connected. This is a contradiction, and the theorem is proved. Theorem IX.1.4 is a special case of Theorem IX.1.3.

IX.2. Relations between isolation of the fractional parts of the exponents and the distribution of singularities

Here are some supplementary definitions. If $Q=\{q_n\}$ and $\Delta=\{\delta_n\}$ are two sequences of positive real numbers, the set $\Delta(Q)$ defined by

$$\Delta[Q] = \bigcup_n (q_n - \delta_n, q_n + \delta_n),$$

is called the *neighbourhood* Δ of Q. Thus,

$$\Delta[(\Lambda)] = \bigcup_n ((\lambda_n) - \delta_n, (\lambda_n) + \delta_n).$$

Let $D^{\cdot}$ be the upper density of Λ and let t_0 be any real number. We denote by $D(\Lambda; t_0)$ the half-strip defined by

$$\text{(IX.22)} \quad |t - t_0| \leqslant \pi D^{\cdot}, \qquad \sigma \geqslant -3[4 - \log D^{\cdot}]D^{\cdot}$$

if $D^{\cdot}>0$; $D(\Lambda; t_0)$ is the half-line $t=t_0$, $\sigma\geqslant 0$, if $D^{\cdot}=0$.

Let S be an arbitrary domain and denote by S^* the set composed of all the points $s+2k\pi i$, where s is any point of S and k any integer. We call S^* the *domain* S *modulo* $2\pi i$.

If l is any real number, then $c[l]$ denotes the set of all the numbers $l+k$, where k is any integer (hence $c[l]$ is the class of numbers congruent to l modulo 1).

A real number a is said to *belong modulo 1 to a set* A if there exists an integer k such that $a+k \in A$. If P is a sequence and $Q \subset P$, then the set of elements of P belonging modulo 1 to $\Delta[Q]$ is said to be *isolated modulo 1 by the neighbourhood* Δ of Q.

We prove the following theorem.

THEOREM IX.2.1. *Suppose that the step of* Λ *is positive and that*

$$f(s) = (A, \Lambda),$$

with $\sigma_a = 0$. *Let* $M = \{\lambda_{n_p}\}$ $(\subset \Lambda = \{\lambda_n\})$, *where for* $\lambda_{n_p} = \mu_p$, $a_{n_p} = b_p$:

$$\text{(IX.23)} \quad \lim \frac{\log |b_n|}{\mu_n} = 0.$$

Let $\Delta = \{\delta_n\}$ *a positive sequence such that* $\lim \delta_n = 0$ *and let* $A(t)$ $(t \geqslant 0)$ *be a positive non-decreasing function satisfying the condition* (IX.1). *We define the function* $h(a)$ $(a > 0)$ *by* (IX.2). *Suppose that the following condition holds:*

$$\text{(IX.24)} \quad h(\mu_n) \leqslant c\delta_n \quad (n \geqslant 1),$$

where c is a constant less than $(2e)^{-1}$.

Then f has no analytic continuation into S^* (*the domain S modulo* $2\pi i$), *where S is an open domain such that*

$$\text{(IX.25)} \quad S \supset D(\Lambda'; t_0),$$

t_0 *being any fixed real number,* Λ' *the set of terms* λ_n *of* Λ *that belong modulo 1 to* $\Delta[(M)]$, *so as to satisfy in* S^* *the condition*

$$\text{(IX.26)} \quad f(s) = O[A(|t|)] \quad (|t| \to \infty),$$

uniformly in σ *as* σ *varies on a compact set.*

Observe that this theorem is non-trivial only if for the upper density $D_1^{\cdot}$ of Λ' we have $D_1^{\cdot} < 1$. For if $1 \leqslant D_1^{\cdot} \leqslant e^4$, then S^* covers, by the definition of $D(\Lambda'; t_0)$ an entire half-plane P_d with $d < 0$, where f cannot be continued *ipso facto* (the step of Λ being positive, σ_a is also the abscissa of holomorphy of f); but if $D_1^{\cdot} > e^4$, then S^*, if it does not contain a half-plane P_d with $d < 0$, itself forms part of the half-plane of the convergence of f, and the question of analytic continuation into S^* does not arise.

Intuitively this theorem can be expressed in the following fashion. For a series $f(s) = (A, \Lambda)$ $(\sigma_a = 0)$ to have an analytic continuation in a half-strip $|t| \leqslant b$, $\sigma > 3(4 - \log b)b$ modulo $2\pi i$ and to grow not faster than $A(|t|)$, it is necessary that every neighbourhood Δ of the fractional parts of a *useful* subsequence M of Λ (that is: $M \subset \Lambda$, $\lim(\log|b_p|/\mu_p)=0$) which isolates modulo 1 as subsequence of Λ whose upper density does not exceed b, should be sufficiently small with respect to the useful terms (that is to say, (IX.24) should not hold).

We proceed to the proof of the theorem.

From M we can extract a sequence $M'=\{\mu_n'\}$ such that if we set

$$\mu_n' = \mu_{m_n} \quad (n \geqslant 1), \qquad \delta_n' = \delta_{m_n}, \qquad \Delta' = \{\delta_n'\},$$

and denote by $\Delta'[(M')]$ the set $\bigcup((\mu_n')-\delta_n', (\mu_n')+\delta_n')$, we have

$$\sup \Delta'[(M')] - \inf \Delta'[(M')] \leqslant \min(1, h_1),$$

where h_1 is the step of Λ'.

The set

$$\Lambda_1' = \{\lambda_n | \lambda_n \in \Lambda,\ c[\lambda_n] \cap \Delta'[M']\} \neq \emptyset$$

is a subsequence of Λ'. It follows, by the definition of the sets D, that $D(\Lambda_1'; t_0) \subset D(\Lambda'; t_0)$. Consequently $S \supset D(\Lambda_1'; t_0)$. Therefore, if we can prove the theorem for the case when M satisfies the additional condition

$$\text{(IX.27)} \quad \sup \Delta[(M)] - \inf \Delta[(M)] < \min(1, h_1),$$

where h_1 is the step of Λ', then the theorem is proved in the general case. Consequently we are going to assume that (IX.27) holds.

We suppose that the series $f(s)=(A, \Lambda)$, can be continued in S^* and can satisfy there the condition $f(s)=O[A(|t|)]$ ($|t| \to \infty$) and we show that this leads to a contradiction.

As in the proof of IX.1.1 we construct the functions $F_a(z)$ of V.4.1 with $C(x)=\log A(x)$ for $x>x_0$, where x_0 is such that $A(x_0) \geqslant 1$, and $C(x)=0$ for[2] $x \leqslant x_0$, the quantities p and q having been chosen as in the proof of Theorem IX.1.1. We set $a=\mu_n$ and write

$$\Phi_n(z) = F_{\mu_n}(z) \quad (n \geqslant 1).$$

We define $\Phi(z)$ by (IX.16), with α_n replaced by (μ_n), the fractional part of μ_n. The inequalities (IX.13), (IX.14) and (IX.15) are now replaced, respectively, by

$$|\Phi_n(z)| \leqslant e^{\delta_n|y|-C(|z|)}\mathfrak{L}_n(x)$$

$$\text{(IX.28)} \quad \hat{\Phi}_n(u) \geqslant g\, e^{-\varepsilon_n'\mu_n} \quad \text{for} \quad |u| \leqslant l\delta_n$$

$$\text{(IX.29)} \quad \hat{\Phi}_n(u) = 0 \qquad\qquad \text{for} \quad |u| \geqslant \delta_n,$$

where $\mathfrak{L}_n$ has the same meaning as in (IX.13), g and l have the same values as in (IX.14) and (IX.15), and $\varepsilon'_n = h(\mu_n)$. We can then verify that the following relations hold:

$$\text{(IX.30)}\quad \Delta[(M)] \supset \text{support of } \hat{\Phi}(u),$$

$$\text{(IX.31)}\quad \lim \frac{\log \hat{\Phi}(\mu_n)}{\mu_n} = 0.$$

If we now set, as in the proof of Theorem IX.1.1, $\Psi(\zeta) = (e^{\zeta} - 1)^{-1}$, we can define $\Psi(s)$ by the integral (IX.18), where Φ has the meaning we have just given to it and f is given in the statement. Since the value of c occurring in (IX.18) is still arbitrary positive, we see that $\Psi(s)$ is holomorphic for $\sigma > 0$ and bounded for $\sigma \geqslant \varepsilon > 0$, where $\varepsilon > 0$ is arbitrary.

There exists an R and an a such that if B denotes the half-strip $|t - t_0| < \pi R$, $\sigma > a$, the following conditions hold:

$$S \supset \bar{B}, \qquad B \supset D(\Lambda'; t_0), \qquad R > D_1^{\cdot}, \qquad a < \sigma_0,$$

where $D_1^{\cdot}$ is the upper density of Λ' and $\sigma_0 = \inf \sigma$ for $s \in D(\Lambda'; t_0)$.

Denoting by D_c the contour of B_c defined by

$$B_c = B^* \cup P_c,$$

we see, by applying the methods already used in this book, that for $\sigma > 0$:

$$\Psi(s) = \int_{D_c} f(z)\psi(s - z)\Phi(-iz)\, dz.$$

If we set

$$0 < \varepsilon < R - D_1^{\cdot}$$

it is easy to see that when s varies in the half-strip $B(\varepsilon)$ defined by $|t - t_0| < \pi(R - \varepsilon)$, $\sigma > a + \pi\varepsilon$, z varying on D_c, then $\psi(s - z)$ is a holomorphic function of s and of z and that it is bounded. It follows that $\Psi(s)$ is holomorphic in $B^*(\varepsilon) \cup P_{c+2\varepsilon}$. Since $\varepsilon > 0$ is arbitrarily small, we conclude that $\Psi(s)$ is holomorphic in $B^* \cup P_c$.

Still imitating the argument we have used in proving Theorem IX.1.1 we obtain

$$\Psi(s) = i \sum e^{-ns} \sum a_m \hat{\Phi}(\lambda_m - n).$$

Now by (IX.30) we know that $\hat{\Phi}(\lambda_m - n) \neq 0$ only when $\lambda_m \in \Lambda'$, and by (IX.27) to every $\lambda_m \in \Lambda'$ there corresponds a single n such that

$$\lambda_m - n \in \Delta[(M)].$$

However, by the same inequality (IX.27), if for sufficiently large n there exists a $\lambda_m \in \Lambda'$ such that $\lambda_m - n \in \Delta[(M)]$, then this λ_m is the only one having this property. We denote by $\{\nu_j\}$ the sequence of such integers n and let λ_{m_j} be the term of Λ' corresponding to ν_j (hence $\lambda_{m_j} - \nu_j \in \Delta[(M)]$). There can only be finitely many integers n such that to each of them there corresponds more than one λ_m with $\lambda_m - n \in \Delta[(M)]$; to each of these finitely many n there correspond only finitely many such λ_m.

Therefore Ψ can be written for $\sigma > 0$ in the following form:

$$\text{(IX.32)} \quad \Psi(s) = \sum_{n \leqslant k} q_n e^{-ns} + i \sum a_{m_j} \hat{\Phi}(\lambda_{m_j} - \nu_j) e^{-\nu_j s},$$

where k is a non-negative finite integer. The series occurring in this expression converges for $\sigma > 0$, because $\Psi(s)$ is holomorphic in this half-plane.

For $m_j = n_p$ we have $(\lambda_{m_j} - \nu_j) = (\lambda_{n_p}) = (\mu_p)$. We also have $a_{m_j} = b_p$. Now

$$\hat{\Phi}(u) = \sum \frac{\hat{\Phi}_p(u - (\mu_p))}{p^2},$$

where the functions $\hat{\Phi}_p$ are all non-negative; therefore

$$\text{(IX.33)} \quad \frac{\hat{\Phi}_p(0)}{p^2} \leqslant \hat{\Phi}((\mu_p)).$$

So we see, by (IX.28), (IX.31) and (IX.33), that for the Taylor-D series

$$A(s) = \sum a_{m_j} \hat{\Phi}(\lambda_{m_j} - \nu_j) e^{-\nu_j s} = \sum d_j e^{-\nu_j s}$$

the abscissa of convergence is zero.

But we have seen that the function $\Psi(s)$ (hence $A(s)$) can be continued analytically in the half-strip

$$|t - t_0| \leqslant \pi D_1^{\cdot}, \qquad \sigma > -3(4 - \log D_1^{\cdot}) D_1^{\cdot},$$

where $D_1^{\cdot}$ is the upper density of $\Lambda' = \{\lambda_{m_j}\}$, hence of $\{\nu_j\}$. This contradicts Theorem VI.2.1 and thus proves our theorem.

IX.3. Variation of the exponents and the analytic continuation

Let Λ and M be two sequences of exponents, d a positive number, and denote the set of all the differences $d_{n,m}=\lambda_n-\mu_m$, with $\lambda_n \in \Lambda$, $\mu_m \in M$, $|\lambda_n-\mu_m|<d$ by $(\Lambda-M)_d$. If $\nu(x)$ is a positive function for $x \geqslant 0$, and Ω a subset of all the $d_{n,m}$, then the set of $d_{n,m}$ contained in

$$\bigcup_{d_{p,q}\in\Omega} (d_{p,q} - \nu(d_{p,q}), d_{p,q} + \nu(d_{p,q}))$$

is called the *ν-neighbourhood* of Ω and denoted by $\nu(\Omega)$. The set of λ_n belonging to Λ such that there exists a $\mu_m \in M$ with $d_{n,m} \in \nu(\Omega)$ is called the *Λ-component of* $\nu(\Omega)$.

If E is a set belonging to the half-plane $\sigma \leqslant 0$, then the set of points that cannot be joined, starting from the half-plane P_0, by a Jordan curve without passing through a point of $\bar{E}$ is called the *envelope* of E and is denoted by E_*; it is the complement of the largest domain belonging to the complement CE that is connected with P_0.

We recall that when $\varphi=(B, M)$ is holomorphic in a domain $D \supset P_0$ and $Q(t)$ is a positive function for $t \geqslant 0$, we say that φ does not grow in D faster than $Q(t)$ if in the complement of the union of the disks $C(s, \varepsilon)$ with $s \in CD$, $\varepsilon>0$ arbitrary, we have $\varphi(s)=O[Q(|t|)]$ $(|t| \to \infty)$ uniformly in σ with σ varying on a compact set (so far we have only used this definition when D was the complement of the singular set of φ in a half-plane).

We also recall that if $M' \subset$ M, we say that M' is a *useful subsequence of M with respect to B* if for $\{\mu_{n_j}\}=M'$ we have

$$\lim (\log |b_{n_j}|/\mu_{n_j}) = \sigma_{a,\varphi}.$$

Let S be a closed set in the half-plane $\sigma \leqslant 0$ and $M' \subset M$. We say that *S cuts the domain $\mathfrak{D}$ modulo* $[M, M', Q]$ if for any domain $D \supset P_0$ for which there exists a function $\varphi=(B, M)$ holomorphic in D that does not grow in D faster than $Q(t)$, and where M' is a useful subsequence of M with respect to B, we have

$$[S + CD]_* \cap \mathfrak{D} \neq \phi.$$

It is evident that if S cuts $\mathfrak{D}$ modulo $[M, M', Q]$, then S also cuts modulo $[M, M', Q]$ every set $\mathfrak{D}_\gamma$ obtained from $\mathfrak{D}$ by the translation $i\gamma$ (where γ is real). To see this it is sufficient to replace b_n (where $B=\{b_n\}$) by $b_n\, e^{-i\gamma\mu_n}$.

By the theorems stated in this section we shall see that if $f=(A, \Lambda)$ and $\varphi=(B, M)$ have analytic continuations that are uniform, respectively, in D_1 and D_2, where these domains contain the half-plane of convergence ($\sigma>0$ for the two series in our statements), then the upper density of the Λ-component of a sufficiently large neighbourhood $\nu(\Lambda'-M')_d$ (where Λ' and M' are, respectively, useful subsequences of Λ and of M) determines an important relation between D_1 and D_2. Here the 'magnitude' of this neighbourhood depends on the functions $P(t)$ and $Q(t)$ which determine the order of growth of f and of φ in their respective domains. In other words, the sequences Λ, M, Λ', M' and the functions $P(t)$, $Q(t)$ impose a 'dispersion law' on $[CD_1+CD_2]_*$ no matter what the functions $f=(A, \Lambda)$ and $\varphi=(B, M)$ are, provided that f and φ can be continued directly starting from P_0 into D_1 and D_2, respectively.

THEOREM IX.3.1. *Let Λ and M be two sequences of positive step. Let D be a domain with $D \supset P_0$, and let $P(t)$ and $Q(t)$ $(t\geqslant 0)$ be two positive non-decreasing functions, where* $\log P(t)$ *is a concave function.*

Suppose that $f=(A, \Lambda)$ does not grow in D faster than $P(t)$.

Let $M' \subset M$ and let Λ' be a useful subsequence of Λ with respect to A. Let $0\leqslant\nu(x)\leqslant\tau<h_M/2$, where h_M is the step of M, and let $d>0$.

If by setting $C(t)=\log[P(t)Q(t)]$ we have for infinitely many λ_n and μ_m, with $(\lambda_n-\mu_m)\in(\Lambda'-M')_d$:

$$\int_{\lambda_n}^{\infty} \frac{C(t)}{t^2} \leqslant c\nu(\lambda_n - \mu_m),$$

with $2ec<1$, then CD cuts the half-strip

$$|t| \leqslant \pi D^{\cdot}, \qquad \sigma > -3[4-\log(hD^{\cdot})]D^{\cdot},$$

modulo $[m, M', Q]$; where h and $D^{\cdot}$ are, respectively, the step and the upper density of the Λ-component of $(\Lambda'+M')_d$.

We do not give here a proof of this proposition. The reader should be guided by the methods used to prove the other propositions of this chapter (and to exploit the fact that $\log P(t)$ is concave in a fashion similar to that in the proof of Theorem V.4.1) and should then be in a position to reconstruct the proof of Theorem IX.3.1.

NOTES

[1] In this proof we also start with the assumption that $\alpha_n \downarrow \alpha^0$, and an easy change of the argument enables us to pass to the case $\alpha_n \uparrow \alpha^0$.

[2] We set $C(x)\equiv 0$ if $A(x)\leqslant 1$ $(x>0)$.

CHAPTER X

BIBLIOGRAPHICAL NOTES

The results of Chapter I are to a very large extent classical, for example, all the theorems concerning various domains of convergence. We note, however, that the upper mean density was introduced by the author in [36], where there is also the first account of our method to calculate the terms of the associated sequence; these values also come out from research of Ostrowski [47], taken up by Bernstein [8].

All the results of Chapter II are due to the author. For the theorem of II.1 we refer to [41], [42], [45]. But the case when all the λ are distinct modulo 1 had already been treated by the author in the older [31], [35]. We add that in our older papers the form itself of the domain (after translating all the present results to the variable s) is much more restrictive than in the more recent papers and in this chapter.

II.2 is essentially devoted to the 'fundamental inequality' on the coefficients of a Dirichlet series: Theorem II.2.1. For series having an abscissa of convergence (which is the case here) this inequality had already been presented by the author in [32] (under a slightly less general form). It was then generalized by the author to series that need not converge, *a priori*, anywhere: 'adherent series', and this has enabled us to apply it to a great many different branches of analysis (problems of moments, weighted approximation, generalized quasi-analyticity, etc.). See [36] for the general result as well as for the rest of the bibliography concerning this subject. Later results permitting the evaluation of the coefficients of the Dirichlet polynomials tending to a function were obtained by Schwartz [54]. Sunyer y Balaguer [55], [56], [57] and [58] has made a deep generalization of the inequality of the author concerning the adherent series in transferring it, with some new ideas on the '*logarithmic precision*' (of the adherence), to Dirichlet polynomials: 'adherent polynomials'. The results II.2.2 and II.2.3 (the more explicit case of II.2.2), which are both corollaries of the fundamental inequality, are utilized a great deal in the rest of the book.

Theorem III.1.1 is a consequence of the results of the first section of II.1. This theorem likewise had been announced by the author some years earlier under a much less general form [31], [35]. In these papers it is supposed that *all* the λ and the unit element are linearly independent. Theorem III.1.2, announced by the author in [45], is a generalization of III.1.3 (which we call here a corollary to Theorem III.1.2) proved by Aronszajn [5], [6] who relied on our theorem in [31] and used the modular function to pass from our theorem in [31] to his own. As in [31], his domain is more restrictive than that of the present text.

Theorem III.2.1, due to Ritt [52], is classical. Theorem III.2.2 was proved in a joint paper by the author and Gergen [30] in a less general form. In the present form it was proved by the author in [36]. The same remarks apply to III.2.3, which was also proved by the author in [36]. So was II.2.4. Theorem III.2.5 concerning the families of functions tending to infinity is due to the author [29]. Theorem III.2.6 is a special case of III.2.8, the Valiron-Bieberbach theorem [59]. The proof given here, which is based on III.2.5, appears for the first time in [36]. Theorems III.2.7 and III.2.9 are due to the author [36].

Chapter IV deals with Taylor series, essentially with Theorem IV.1.1, published by the author in 1937 [33], and to some of its applications. This Theorem IV.1.1 has been generalized by many authors, among them Denjoy [15], Lévy [23], Perron [48], Bouligand [12], von Mises [46], Dvoretzky [16], [17], Bergman [7], to several variables, etc. Theorems IV.1.5 and IV.1.6, which are both applications of Theorem IV.1.1, are due to the author, the first published in [38], the second in [34]. Theorem IV.1.6 was taken up and generalized by Valiron [60].

Theorems V.1.1, V.2.1, the subsequent lemmas, and Theorem V.2.5 are due to the author [27]. The integral (V.3) introduced by the author has enabled Widder [61] (on the advice of the author) to establish a particular case of V.2.1. Theorem V.1.1 was generalized by Bochner [10] who states it for general almost periodic functions. V.2.7 is a particular case of a theorem of Bernstein [8]. Theorem V.2.8 is due to the author [27], [35]. The 'fictitious composition' of V.3 and the relevant Theorem V.3.1 are due to the author [28]. Brunk [13] has proved some interesting results on such a composition. Blambert has indicated conditions for a 'composite' point to be necessarily singular [9].

The functions F_a of Chapter V with their properties expressed by V.4.1 were introduced by the author [39] (see also [44]). The version given in V.4.1 is more precise than that of [39] and of [44]. The precise form of the inequalities concerning these functions as well as their Fourier trans-forms, which is expressed by the relations (V.16), (V.17) and (V.18) – the last one an evident consequence of (V.16) by the Paley-Wiener theorem – play an essential role in Chapter IX, above all in the applications of V.4.1. The existence of a function F satisfying, together with its Fourier transform, less precise inequalities have been demonstrated by Levinson [22] and Agmon [2]. V.4.2 is due to the author [39]. Theorem V.4.3 is due to Agmon (stated under a slightly different form). Theorem V.4.4 is also due to Agmon [2]. The results of V.5 (Theorems V.5.1, V.5.2, and V.5.3) are due to the author [27], [35]. Theorem V.5.1 is more precise than the corresponding theorem in [35]. Schottländer [53] has made an interesting synthesis of the composition theorems, mainly for Taylor series.

Theorems VI.1.1 and VI.1.2 have not been published elsewhere.

In contrast, an old theorem of the author [26] and another one given (under a slightly less general form) by Aronszajn [6] can be regarded as very closely related to Theorem VI.1.3.

Theorem VI.2.1 is due to the author [35]. This theorem contains as a special case a theorem of Ostrowski [47] according to which the function cannot remain holomorphic on a disk with centre on the axis of convergence, of radius only slightly different from $A(h, D^{\cdot})$.

Theorem VI.2.3 has not been published elsewhere.

The Theorems VI.3.1, VI.3.2, and VI.3.3 are due to the author [35]. For Taylor series (or rather Taylor-D, to stick to our notation) Theorems VI.3.1 and VI.3.2 were proved by Pólya [51]. But in Pólya's theorems the maximum density occurs, a quantity not inferior to $D^{\cdot}$ (which can, however, be equal to infinity when $D^{\cdot}$ is finite).

Theorems VI.4.1 is the classical theorem of Cramèr (see [8]). Theorem VI.4.2 is due to Pólya [49], [51] (Theorem of Cramèr-Pólya) and Theorem VI.4.3 is due to Bernstein [8].

Theorem VI.4.4 has not yet been published. It generalizes Theorem VI.4.5, due to the author [24]. Theorem VI.4.5 has been generalized by Pólya [50] and Jungen [20] who replace the word 'pole' by singular algebraic-logarithmic point. Theorem VI.4.6 is due to the author (for $N \geqslant 2$ see [25] for $N = 1$ see [43]). Theorem VI.4.7 and its corollary VI.4.8 are due to the author [43].

The theorems of VII.1 are due to Agmon [1], [3]. As to Theorem VII.2.1, it forms a modification of a theorem of Agmon [4]. In the form in which it is stated (apart from the conditions (VII.20) and (VII.21) which are satisfied *ipso facto*, owing to the other conditions of the theorem, as we have explained in the remark following this statement) it can be found in a common paper by the author and Chandrasekharan [14]. The given modification of the original theorem of Agmon appeared interesting for the applications to the generalized Riemann equation set forth in Chapter VIII. Incidentally, it is Theorem III.2.3 that permits the passage from Agmon's theorem to the theorem stated.

Chapter VIII finds its place in this monograph mainly to illustrate some new applications of Theorem III.2.3, which transfers Theorem II.2.1 to a strip.

Lemma VIII.1.1 is essentially due to Bochner and Chandrasekharan [11]. The form given here to this lemma is that of [21], which was communicated to one of the authors of this paper by Chandrasekharan.

Theorem VIII.1.2 was published in a joint paper of the author and Chandrasekharan [14]. It improves the conditions of a theorem of Bochner and Chandrasekharan [11]. Theorem VIII.1.3 is also in [14], as well as the corollaries following Theorem VIII.1.3. We remark that the conditions (VIII.19) and (VIII.20) do not occur in the original paper, because they are superfluous, just like the conditions (VII.20) and (VII.21) in Theorems VII.2.1.

All the results of Chapter IX are due to the author. The account in this monograph has less restrictive and simpler conditions than in the original papers of the author. Thus, condition (IX.3), which occurs in Theorem IX.1.1, is much simpler than the corresponding condition in [37]. The same paper also contains Theorems IX.1.2, IX.1.3, and IX.1.4. Theorem IX.2.1, which is certainly the most important in this chapter, was proved under less general conditions in [39].

Theorem IX.3.1, for which we do not give a proof in the text, hoping that the reader will easily be able to reconstruct it if he has had the patience to read and to study the proofs of the preceding theorems of this chapter, is published in a slightly more complicated form in [40]. This paper also contains several other theorems on the same subject, as well as new applications to the generalized Riemann equation.

NOTE

ON THE DIRICHLET SERIES REPRESENTING MEROMORPHIC FUNCTIONS

Let $f=(A, \Lambda)$ and let $\sigma_1<\sigma_a=\sigma_{a,f}$. If $s' \in P_{\sigma_1}$, we define the *class* $E_f(s')$ of $S_{\sigma_1,f}$ or simply, *class* $E(s')$, if no ambiguity is to be feared, as the set of all the points of $S_{\sigma_1,f}$ that can be put in the form $s'+2k\pi i$, where k is an integer.

If $E(s')$ contains only poles, we say that this class is *polar*. If the orders of the poles of $E(s')$ have a finite maximum r, we say that $E(s')$ is *polar of order* r.

If $\alpha=s'+2k\pi i$ is a point of a polar class of order r, we denote by $A_1^{(k)}, \ldots, A_r^{(k)}(s')$, respectively, the coefficients of

$$(s-\alpha)^{-1}, \ldots, (s-\alpha)^{-r}$$

of the principal part of this pole. Then $A_{q+1}^{(k)}(s')=\cdots=A_r^{(k)}(s')=0$ if $q<r$, where α is a pole of order q. If for an integer k we have $s'+2k\pi i \notin E(s')$, we set $A_j^{(k)}(s')=0$ for $1 \leqslant j \leqslant r$. We set

$$\rho(s') = \sup_j \limsup_{|k| \to \infty} \frac{\log |A_j^{(k)}(s')|}{\log |k|}.$$

If f has in P_{σ_1} no singularities other than poles belonging to the classes $E(s^{(1)}), \ldots, E(s^{(m)}), \ldots$ and if f is order ν in P_{σ_1}, except at the singularities, we say that f is *meromorphic in* P_{σ_1} and that *the polar order of f in P_{σ_1} at infinity is p, defined by*

$$p = \sup (\nu, p(s^{(1)}), p(s^{(2)}), \ldots, p(s^{(m)}), \ldots).$$

In what follows we assume that $\sigma_1<\sigma_a=\sigma_{a,f}=0$.

The following theorem holds:[1]

THEOREM 1. (1) *Suppose that $f=(A, \Lambda)$ is uniform and meromorphic of finite order in P_{σ_1} at infinity, and that f has in this half-plane only finitely many classes $E(s^{(1)}), \ldots, E(s^{(l)})$. Suppose that the classes $E(s^{(1)}), \ldots, E(s^{(m)})$ $(m<l)$ are situated on $\sigma=0$ and that $r_1, \ldots, r_m$, respectively, are their orders. We set $s^{(j)}=\sigma^{(j)}+it^{(j)}$, and $\sigma_P=\sup_{j \geqslant m} \sigma^{(j)}$.*

(2) *Let p be the polar order at infinity of f in P_{σ_1}.*

We denote by $C_n^{(g)}$ *the n-*th *Taylor coefficient of f of (integral) order* $g>p$.

We set

$$N = \sum_{j \leqslant m} r_j + g + 1$$

if $s=0$ *is a pole of f or a regular point of f with* $f(0) \neq 0$*; and we set*

$$N = \sum_{j \leqslant m} r_j + \max(g + 1 - h, 0)$$

if f is regular at $s=0$ *and this point is a zero of positive order h. Setting*

$$D_{n,N} = \begin{vmatrix} C_n^{(g)} & \dots & C_{n+N}^{(g)} \\ C_{n+N}^{(g)} & \dots & C_{n+2N}^{(g)} \end{vmatrix}$$

we have

$$\limsup_{n \to \infty} \frac{\log |D_{n,N}|}{n} \leqslant \sigma_p.$$

If all the classes $E(s^{(1)}), E(s^{(l)}), \dots, E(s^{(l)})$ *are situated on* $\sigma=0$ $(m=l)$, *then*

$$\limsup_{n \to \infty} \frac{\log |D_{n,N}|}{n} \leqslant \sigma_1.$$

Proof. Let $c>0$; we set

$$\varphi(s) = (e^s - 1)^{-1},$$

and consider the integral

$$F(z) = \frac{g!}{2\pi} \int_{c-i\infty}^{c+i\infty} f(s)\varphi(z - s) \frac{dt}{s^{g+1}} \quad (z = x + iy,\ s = \sigma + it) \tag{1}$$

for $x>c$. For these values of z:

$$F(z) = \frac{g!}{2\pi i} \sum \sum \int_{C_\varepsilon^{k,q}} f(s)\varphi(z - s) \frac{ds}{s^{g+1}} + \frac{g!}{2\pi i} \int_{\sigma_1+\varepsilon-i\infty}^{\sigma_1+\varepsilon+i\infty} f(s)\varphi(z - s) \frac{ds}{s^{g+1}}, \tag{2}$$

the double sum being extended over $1 \leqslant q \leqslant l$, where k is an arbitrary integer and over $q=0$, $k=0$ if $E(0)$ is not a polar class; $C_\varepsilon^{0,0}$ is the circle $|s|=\varepsilon$, $C_\varepsilon^{k,q}$, $1 \leqslant q \leqslant l$, is the circle $|s-s^{(q)}-2k\pi i|=\varepsilon$, where $\varepsilon>0$ is chosen so small that for each of closed disks bounded by $C_\varepsilon^{k,q}$ contains only the point $s^{(q)}+2k\pi i$ if $q\geqslant 1$, and only $s=0$ if $q=0$, and where all these closed disks are contained in $P_{\sigma_1+\varepsilon}$ and in the half-plane $x<c-\varepsilon$.

If $s^{(q)}+2k\pi i$ does not belong, for a given k, to $E(s^{(q)})$, then

$$\int_{C_\varepsilon^{k,q}} f(s)\varphi(z-s)\frac{ds}{s^{g+1}}=0.$$

On the other hand, it is obvious that the function of z represented by the last integral in (2) is holomorphic for $x>\sigma_1+\varepsilon$.

Thus, $F(z)$ can be put for $x>c$ in the form

$$F(z)=\frac{g!}{2\pi i}\sum\sum\int_{C_\varepsilon^{k,q}} f(s)\frac{ds}{(e^{z-s}-1)s^{g+1}}+T(z),$$

where $T(z)$ is holomorphic in the half-plane $x>\sigma_1+\varepsilon$.

Since by assumption the polar order of f at infinity in P_{σ_1} is less than g, we see that the sum

$$F_q(z)=\sum_{|k|\geqslant 0}\int_{C_\varepsilon^{k,q}} f(s)\varphi(z-s)\frac{ds}{s^{g+1}}$$

converges absolutely and uniformly on every compact set when x is sufficiently large. If r_q is the order of the polar class $E(s^{(q)})$, which is not the class $E(0)$ (if such a class exists!), we find that $F_q(z)$ is a function that has as singularities only the points $s^{(q)}+2k\pi i$ in P_{σ_1}, which are poles of order at most r_q. However, the points $2k\pi i$ can be poles of order r_q+g+1 if $E(0)$ is a polar class of order r_q, or of order $g+1$ if $f(s)$ is regular at $s=0$.

On the other hand, the integral in (1) can be represented for sufficiently large x as $(2\pi/g!)\sum C_n^{(g)}e^{-nz}$.

Therefore this Taylor-D series represents a meromorphic function in P_{σ_1} which has there at most N poles.

By Theorem V.2.1 the real part of the places of these poles in the half-planes $\sigma<0$ does not exceed σ_P. It is now enough to apply Hadamard's theorem on the polar singularities of a Taylor series to arrive at the required result.

THEOREM 2. *Suppose that the sequence Λ satisfies the condition*

$$|\lambda_n-n|\leqslant a<\tfrac{1}{2}\quad(n\geqslant 1),$$

that condition (1) *of Theorem* 1 *holds, and that condition* (2) *of Theorem* 1 *is replaced by the following condition:*

(3) $f(s)$ *does not grow in* P_{σ_1}, *except at the singularities, faster than* $\exp C(|t|)$, *where the non-decreasing function* $C(t)$ $(t>0)$ *satisfies the condition*

$$\int_1^\infty \frac{C(t)}{t^2}\,dt < \infty.$$

By setting

$$N = \sum_{1 \leqslant j \leqslant m} r_j,$$

and writing

$$D_{n,N} = \begin{vmatrix} a_n & \cdots & a_{n+N} \\ \vdots & & \\ a_{n+N} & \cdots & a_{n+2N} \end{vmatrix}$$

we have

$$\limsup \frac{\log |D_{n,N}|}{n} \leqslant \sigma_P.$$

To prove Theorem 2 we utilize instead of (1) the expression

$$F(z) = \int_{c-i\infty}^{c+i\infty} f(s)\varphi(z-s)H(-is)\,ds,$$

where $\varphi(s)$ is the function defined in the proof of Theorem 1 and where $H(z)$ is defined as in the proof of Theorem V.4.3, with $C_1(u)$ replaced by $C(u)+a\log u$; $a > \sup(\rho(s^{(1)}), \ldots, \rho(s^{(l)}))$ and $C_2(u) \equiv 0$, $\{\mu_n\}$ being the sequence of positive integers. We imitate the proof of Theorem V.4.3 and that of Theorem 1 of this note and finish up again by the application of Hadamard's theorem that was used in the proof of Theorem 1.

NOTE

[1] This theorem was proved in a more restrictive form in [27].

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